Balls of Permutations in the $\ell_\infty$-Metric

Moshe Schwartz

Electrical and Computer Engineering
Ben-Gurion University of the Negev
An age of data

Data storage is growing exponentially.

Storage technologies are crucial to enabling big data and cloud paradigms.

What storage options do we have?

Magnetic HDD
- Cheap
- Many R/W operations

Flash SSD
- Fast
- Energy Efficient
- Resilient (no mechanical parts)

ℓ∞-Metric Permutation Balls

Introduction
What storage options do we have?

<table>
<thead>
<tr>
<th>Magnetic HDD</th>
<th>Flash SSD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cheap</td>
<td>Fast</td>
</tr>
<tr>
<td>Many R/W operations</td>
<td>Energy Efficient</td>
</tr>
<tr>
<td></td>
<td>Resilient (no mechanical parts)</td>
</tr>
</tbody>
</table>
A flash memory cell is a bucket

Flash memory is an electronic non-volatile memory.

Conventional multi-level flash technology allows each cell to store $q$ discrete values. These are represented by appropriate charge-level intervals.
Flash is an asymmetric noisy bucket

1. Adding charge is much easier than removing it. (asymmetric)
2. Errors commonly change charge level by a limited magnitude. (noisy)

Adding charge is much easier than removing it.

Errors commonly change charge level by a limited magnitude.
Flash is an asymmetric noisy bucket

1 Adding charge is much easier than removing it. (asymmetric)

2 Errors commonly change charge level by a limited magnitude. (noisy)

1 Writing is slow.

2 Data loss occurs with time.
Cells are compared against each other.

Information is stored in the permutation induced by their ranking.
Cells are compared against each other.

Information is stored in the permutation induced by their ranking.

1. Faster writing
2. Less errors.

The goal

Construct Error-Correcting Codes for Rank Modulation
What do errors look like?

Limited-magnitude change in charge level

\[ \downarrow \]

Limited-magnitude change in ranks of cells

\[ \downarrow \]

Limited $\ell_\infty$-distance between stored permutations

**Definition**

For all $f, g \in S_n$, $d_\infty(f, g) = \max_i |f(i) - g(i)|$. 

$\ell_\infty$-Metric Permutation Balls  
Introduction  
8 / 38
A simple code

$(S_n, d_\infty)$ is a metric space, so we can define error-correcting codes.

**Definition**

An $(n, M, d)$-LMRM code (Limited-Magnitude Rank-Modulation code) is a subset $C \subseteq S_n$ of size $M$ such that $d_\infty(f, g) \geq d$ for all $f, g \in C, f \neq g$.

*Tamo and Schwartz, IEEE Trans. Inform. Th., 2010*

*Kløve, Lin, Tsai, and Tzeng, IEEE Trans. Inform. Th., 2010*
A simple code

$(S_n, d_\infty)$ is a metric space, so we can define error-correcting codes.

**Definition**

An $(n, M, d)$-LMRM code (Limited-Magnitude Rank-Modulation code) is a subset $C \subseteq S_n$ of size $M$ such that $d_\infty(f, g) \geq d$ for all $f, g \in C, f \neq g$.

**Construction**

Given $n, d \in \mathbb{N}$ we construct

$$C = \{f \in S_n \mid \forall i \in [n], f(i) \equiv i \pmod{d}\}.$$
How good is the code?

We are interested in the asymptotics of the code. For an \((n, M, d)\)-LMRM code we define the normalized distance and rate:

\[
\delta = \frac{d}{n - 1}, \quad R = \frac{\log_2 M}{n}.
\]
How good is the code?

We are interested in the asymptotics of the code. For an \((n, M, d)\)-LMRM code we define the normalized distance and rate:

\[ \delta = \frac{d}{n - 1}, \quad R = \frac{\log_2 M}{n}. \]

**Theorem**

For \(0 < \delta \leq 1\), the construction produces codes of rate

\[ R = \left(1 - \delta \left\lfloor \frac{1}{\delta} \right\rfloor\right) \log_2 \left(\left\lceil \frac{1}{\delta} \right\rceil!\right) + \left(\delta + \delta \left\lfloor \frac{1}{\delta} \right\rfloor - 1\right) \log_2 \left(\left\lfloor \frac{1}{\delta} \right\rfloor!\right) + o(1). \]
How good is the code?

The code

$R$ vs $\delta$

$\ell_\infty$-Metric Permutation Balls

Introduction
How good is the code? (Cont.)

The code

Lower bound on [LTT'08] construction

Upper bound on [LTT'08] construction

Lin, Tsai, and Tzeng, Int. Symp. on Inform. Th., 2008
We can obtain better bounds

- A **ball** of radius $r$ centered at $f$ is $B_{r,n}(f) = \{g \in S_n \mid d_{\infty}(f,g) \leq r\}$.
We can obtain better bounds

- A ball of radius $r$ centered at $f$ is $B_{r,n}(f) = \{g \in S_n \mid d_\infty(f, g) \leq r\}$.
- The $\ell_\infty$-metric is right invariant, i.e., for all $f, g, h \in S_n$, we have $d_\infty(f, g) = d_\infty(fh, gh)$. 

Theorem (Ball-Packing Bound)

Let $C$ be an $(n; M; d)$-LMRM code. Then $B_{\lceil (d_\infty)/2 \rceil; n} M \leq n!$. 

Theorem (Gilbert-Varshamov-Like Bound)

Let $n, M, d, g$ be positive integers such that $B_{d; n} M \leq n!$. Then there exists an $(n; M; d)$-LMRM code.
We can obtain better bounds

- A ball of radius $r$ centered at $f$ is $B_{r,n}(f) = \{g \in S_n \mid d_\infty(f, g) \leq r\}$.
- The $\ell_\infty$-metric is right invariant, i.e., for all $f, g, h \in S_n$, we have $d_\infty(f, g) = d_\infty(fh, gh)$.
- Thus, ball sizes depend only on the radius and are denoted $|B_{r,n}|$. 
We can obtain better bounds

- A ball of radius $r$ centered at $f$ is $B_{r,n}(f) = \{ g \in S_n \mid d_\infty(f, g) \leq r \}$.
- The $\ell_\infty$-metric is right invariant, i.e., for all $f, g, h \in S_n$, we have $d_\infty(f, g) = d_\infty(fh, gh)$.
- Thus, ball sizes depend only on the radius and are denoted $|B_{r,n}|$.

**Theorem (Gilbert-Varshamov-Like Bound)**

Let $n$, $M$, and $d$, be positive integers such that $|B_{d-1,n}| M \leq n!$. Then there exists an $(n, M, d)$-LMRM code.

**Theorem (Ball-Packing Bound)**

Let $C$ be an $(n, M, d)$-LMRM code. Then $|B_{\lceil(d-1)/2\rceil,n}| M \leq n!$.
But wait...

What is the size of a ball?
Permanants count certain permutation sets

**Definition**

For an $n \times n$ binary matrix $A = (a_{i,j})$, the **permanent** of $A$ is defined as

$$\text{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$
Permanants count certain permutation sets

**Definition**

For an $n \times n$ binary matrix $A = (a_{i,j})$, the **permanent** of $A$ is defined as

$$\text{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$ 

**Example**

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}$$
Permanants count certain permutation sets

**Definition**

For an $n \times n$ binary matrix $A = (a_{i,j})$, the **permanent** of $A$ is defined as

$$\text{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$  

**Example**

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}, \quad f = [2, 1, 4, 3, 5] \text{ contributes 1 to } \text{per}(A).
\]
Permanent count certain permutation sets

**Definition**

For an \( n \times n \) binary matrix \( A = (a_{i,j}) \), the **permanent** of \( A \) is defined as

\[
per(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i, f(i)}.
\]

**Example**

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix},
\]

\( f = [1, 4, 5, 3, 2] \) does not contribute to \( per(A) \).
The size of a ball is a permanent

Let us define the following family of $n \times n$ matrices, $A_{r,n} = (a_{i,j}^{r,n})$, where

$$a_{i,j}^{r,n} = \begin{cases} 
1 & |i - j| \leq r, \\
0 & \text{otherwise.}
\end{cases}$$

**Theorem**

$$|B_{r,n}| = \text{per}(A_{r,n}).$$

**Proof.**

$f \in S_n$ contributes to $\text{per}(A_{r,n})$ iff $|f(i) - i| \leq r$ for all $i$, which holds iff $d_\infty(f, \text{Id}) \leq r$. $\square$
Off-the-shelf bounds on permanents exist

**Theorem (Brégman's Theorem)**

Let $A$ be an $n \times n$ binary matrix, with $r_i$ 1's in the $i$th row. Then

$$\text{per}(A) \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}.$$  

**Theorem (Van der Waerden's Theorem)**

Let $A$ be an $n \times n$ doubly-stochastic matrix, then

$$\text{per}(A) \geq \frac{n!}{n^n}.$$  


Brégman's Theorem gives an immediate upper bound:

\[ |B_{r,n}| = \text{per}(A_{r,n}) \leq \begin{cases} 
((2r + 1)!)^{\frac{n-2r}{2r+1}} \prod_{i=r+1}^{2r} (i!)^{\frac{2}{i}} & 0 \leq r \leq \frac{n-1}{2}, \\
(n!)^{\frac{2r+2-n}{n}} \prod_{i=r+1}^{n-1} (i!)^{\frac{2}{i}} & \frac{n-1}{2} \leq r \leq n - 1.
\end{cases} \]

Van der Waerden's Theorem gives a lower bound:

\[ |B_{r,n}| = \text{per}(A_{r,n}) \geq \begin{cases} 
\frac{n!(2r+1)^n}{2^{2r} n^n} & 0 \leq r \leq \frac{n-1}{2}, \\
\frac{n!}{2^{2(n-r-1)}} & \frac{n-1}{2} \leq r \leq n - 1.
\end{cases} \]

**Proof Sketch**

\[
\begin{array}{c}
\text{per} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \frac{1}{2^{2r}} \text{ per} \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix} \geq \frac{1}{2^{2r}} \text{ per} \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\end{array}
\]
Adding the bounds to the picture

1. The code
2. Lower bound on [LTT'08] construction
3. Upper bound on [LTT'08] construction
4. GV lower bound on optimum code rate
5. Ball-packing upper bound on optimum code rate
A wide gap is the cause (in part)

Rewriting the bounds in asymptotic forms, we get

\[ |B_{\rho(n-1),n}| \leq \begin{cases} \left( \frac{2\rho n}{e} \right)^n \cdot \left( \frac{2}{e} \right)^{2\rho n} \cdot 2^{o(n)} & 0 \leq \rho \leq \frac{1}{2} \\ \frac{n^n}{e^n(3-2\rho)\rho^{2\rho n}} \cdot 2^{o(n)} & \frac{1}{2} \leq \rho \leq 1 \end{cases} := \Phi(\rho, n) \]

\[ |B_{\rho(n-1),n}| \geq \begin{cases} \frac{(2\rho n)^n}{2^{2\rho n}e^n} \cdot 2^{o(n)} & 0 \leq \rho \leq \frac{1}{2} \\ \frac{n^n}{e^n2^{2n(1-\rho)}} \cdot 2^{o(n)} & \frac{1}{2} \leq \rho \leq 1 \end{cases} := \phi(\rho, n) \]
A wide gap is the cause (in part)

Rewriting the bounds in asymptotic forms, we get

\[
\left| B_{\rho(n-1),n} \right| \leq \begin{cases} 
\left( \frac{2\rho n}{e} \right)^n \cdot \left( \frac{2}{e} \right)^{2\rho n} \cdot 2^{o(n)} & 0 \leq \rho \leq \frac{1}{2} \\
\frac{n^n}{en(3-2\rho)} \cdot 2^{o(n)} & \frac{1}{2} \leq \rho \leq 1
\end{cases} := \Phi(\rho, n)
\]

\[
\left| B_{\rho(n-1),n} \right| \geq \begin{cases} 
\left( \frac{2\rho n}{2e^{2\rho n}} \right) \cdot 2^{o(n)} & 0 \leq \rho \leq \frac{1}{2} \\
\frac{n^n}{en^{2n(1-\rho)}} \cdot 2^{o(n)} & \frac{1}{2} \leq \rho \leq 1
\end{cases} := \phi(\rho, n)
\]

The asymptotic gap is therefore,

\[
\limsup_{n \to \infty} \frac{1}{n} \log_2 \frac{\Phi(\rho, n)}{\phi(\rho, n)} = \begin{cases} 
(4 - 2 \log_2 e)\rho & 0 \leq \rho \leq \frac{1}{2}, \\
2(1 - \rho)(1 - \log_2 e) - 2\rho \log_2 \rho & \frac{1}{2} \leq \rho \leq 1.
\end{cases}
\]
A wide gap is the cause (in part)

Asymptotic gap for off-the-shelf ball-size bounds
We can have zero gap for constant radii

The strategy

1. Show that permutations in a ball centered at the identity form a regular language.
2. Use Perron-Frobenius Theory to find the asymptotic size of the regular language.

Lehmer, Comb. Th. and its Applications, 1970
Schwartz, Linear Algebra and its Applications, 2009
We can have zero gap for constant radii

The strategy

1. Show that permutations in a ball centered at the identity form a regular language.
2. Use Perron-Frobenius Theory to find the asymptotic size of the regular language.

- The method is more general.
- For ease of presentation, only \((0, 1)\)-matrices will be shown.
Iteratively building permutations in $B_{r,n}$

**Example:** For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
<th>$f(5)$</th>
<th>$f(6)$</th>
<th>$f(7)$</th>
<th>$f(8)$</th>
<th>$f(9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- At step $i < n$, we set $f(i)$ by choosing a value from $i + [r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b_2 f_0; 1; 0; r$ indexed by $[r, r - 1]$.
- At step $i$, $b_j = 1$ iff $i + j$ was already assigned.
- Moving from step $i$ to step $i + 1$, we drop the left bit, and append a 0 rightmost bit.
Iteratively building permutations in $B_{r,n}$

**Example:** For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
<th>$f(5)$</th>
<th>$f(6)$</th>
<th>$f(7)$</th>
<th>$f(8)$</th>
<th>$f(9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$b_{-2}$</th>
<th>$b_{-1}$</th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Iteratively building permutations in $B_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
<th>$f(5)$</th>
<th>$f(6)$</th>
<th>$f(7)$</th>
<th>$f(8)$</th>
<th>$f(9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$b_{-2} b_{-1} b_0 b_1 | b_2$

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$. 

---

$l_\infty$-Metric Permutation Balls

The $r = O(1)$ Regime
Iteratively building permutations in $B_{r,n}$

**Example:** For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
<th>$f(5)$</th>
<th>$f(6)$</th>
<th>$f(7)$</th>
<th>$f(8)$</th>
<th>$f(9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccc|c}
  b_{-2} & b_{-1} & b_0 & b_1 & b_2 \\
  1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
Iteratively building permutations in $\mathcal{B}_{r,n}$

**Example:** For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th></th>
<th>f(1)</th>
<th>f(2)</th>
<th>f(3)</th>
<th>f(4)</th>
<th>f(5)</th>
<th>f(6)</th>
<th>f(7)</th>
<th>f(8)</th>
<th>f(9)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

```
<table>
<thead>
<tr>
<th></th>
<th>b_{-2}</th>
<th>b_{-1}</th>
<th>b_0</th>
<th>b_1</th>
<th>b_2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
```

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b \in \{0, 1\}^{2r}$ indexed by $[-r, r - 1]$. 

\[ \ell_\infty\text{-Metric Permutation Balls} \]

The $r = O(1)$ Regime
Iteratively building permutations in $B_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th></th>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
<th>$f(5)$</th>
<th>$f(6)$</th>
<th>$f(7)$</th>
<th>$f(8)$</th>
<th>$f(9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$b_{-2}$</th>
<th>$b_{-1}$</th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b \in \{0, 1\}^{2r}$ indexed by $[-r, r - 1]$.
- At step $i$, $b_j = 1$ iff $i + j$ was already assigned.
Iteratively building permutations in $B_{r,n}$

**Example:** For $n = 9$, $r = 2$, $i = 5$, the current status is

<table>
<thead>
<tr>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
<th>$f(5)$</th>
<th>$f(6)$</th>
<th>$f(7)$</th>
<th>$f(8)$</th>
<th>$f(9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccc|c}
  b_{-2} & b_{-1} & b_0 & b_1 & b_2 \\
  1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b \in \{0, 1\}^{2r}$ indexed by $[-r, r - 1]$.
- At step $i$, $b_j = 1$ iff $i + j$ was already assigned.
- Moving from step $i$ to step $i + 1$ we drop the left bit, and append a 0 rightmost bit.
For a binary $b = b_1 b_2 \ldots b_m \in \{0, 1\}^m$:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b0$</td>
<td>Appending a 0 to the end of $b$</td>
</tr>
<tr>
<td>$wt(b)$</td>
<td>The number of non-zero entries in $b$</td>
</tr>
<tr>
<td>$L(b)$</td>
<td>Erasing the leftmost bit, i.e., $b_2 \ldots, b_m$</td>
</tr>
<tr>
<td>$e_k$</td>
<td>The $k$th standard unit vector</td>
</tr>
</tbody>
</table>
We now build a finite state machine

### Construction

We construct a digraph $G_{[-r,r]} = (V, E)$ in the following way:

$$V = \left\{ b = b_{-r}b_{-r+1} \ldots b_{r-1} \in \{0, 1\}^{2r} \mid \text{wt}(b) = r \right\},$$

$$E = \left\{ b \xrightarrow{k} L(b0 + e_k) \mid b, L(b0 + e_k) \in V, \quad k \in T, \quad (b0)_k = 0 \right\}.$$
We now build a finite state machine

Construction

We construct a digraph $G_{[-r,r]} = (V, E)$ in the following way:

$$V = \left\{ b = b_{-r}b_{-r+1} \ldots b_{r-1} \in \{0, 1\}^{2r} \mid \text{wt}(b) = r \right\},$$

$$E = \left\{ b \xrightarrow{k} L(b0 + e_k) \mid b, L(b0 + e_k) \in V, \ k \in T, \ (b0)_k = 0 \right\}.$$

Theorem

There is a bijection between the set of permutations $B_{r,n}$, and the paths of length $n$ in $G_{[-r,r]}$ starting and ending in the vertex $1^r0^r$. 

$\ell_\infty$-Metric Permutation Balls

The $r = O(1)$ Regime
Theorem

For a constant $r$,

$$\lim_{n \to \infty} \frac{1}{n} \log_2 |B_{r,n}| = \log_2 \lambda(A(G_{[-r,r]})),$$

where $A(G_{[-r,r]})$ is the adjacency matrix of $G_{[-r,r]}$, and $\lambda(A(G_{[-r,r]}))$ is the spectral radius of $A(G_{[-r,r]})$. 
Exact asymptotic size

Theorem

For a constant $r$,

$$\lim_{n \to \infty} \frac{1}{n} \log_2 |B_{r,n}| = \log_2 \lambda(A(G_{[-r,r]})),$$

where $A(G_{[-r,r]})$ is the adjacency matrix of $G_{[-r,r]}$, and $\lambda(A(G_{[-r,r]}))$ is the spectral radius of $A(G_{[-r,r]})$.

Proof.

By the previous theorem, $|B_{r,n}| = (A(G_{[-r,r]})^n)_{1,1}$. The matrix $A(G_{[-r,r]})$ has non-negative entries, and can be shown to be primitive. The claim follows immediately by Perron-Frobenius.
Taking radius 2 as an example

Taking $\lambda$ to be the largest (real) root of

$$\chi_A(G_{[-2,2]})(\lambda) = \lambda^6 - \lambda^5 - 2\lambda^4 - 2\lambda^3 - 2\lambda^2 + \lambda + 1,$$

we get

$$|B_{2,n}| = \lambda^{n(1+o(1))} \approx 2.34148^n.$$
What about $r = \Theta(n)$?

**Theorem**

Let $M = (m_{i,j})$ be a $n \times n$ matrix with non-negative entries and $\text{per}(M) > 0$. Additionally, we require that there are two diagonal matrices $D$ and $D'$ with positive diagonal elements such that $D \cdot M \cdot D'$ is a doubly-stochastic matrix. Let $Q = (q_{i,j})$ be an $n \times n$ doubly-stochastic matrix. Then

$$
\log_2 \text{per}(M) \geq \log_2 \frac{n!}{n^n} + \sum_{i,j \in [n]} \left( -q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}} \right).
$$

*Sinkhorn, Ann. Math. Stat., 1964*
Using the Sinkhorn balancing

Proof.

\[
\log_2 \text{per}(M) = \log_2 \text{per}(D \cdot M \cdot D') - \sum_{i \in [n]} \log_2(d_{i,i}) - \sum_{j \in [n]} \log_2(d'_{j,j})
\]

\[
\geq \log_2 \frac{n!}{n^n} - \sum_{i \in [n]} \log_2(d_{i,i}) - \sum_{j \in [n]} \log_2(d'_{j,j})
\]

\[
\geq \log_2 \frac{n!}{n^n} - \sum_{i, j \in [n]} q_{i,j} \log_2 \frac{q_{i,j}}{d_{i,i} \cdot m_{i,j} \cdot d'_{j,j}}
\]

\[
= \log_2 \frac{n!}{n^n} - \sum_{i, j \in [n]} q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}} \quad \square
\]
We can find the optimal matrix

**Lemma**

Fix $\frac{n-1}{2} \leq r \leq n-2$. The optimal $Q = (q_{i,j})$ for $A_{r,n}$ is given by

$$q_{i,j} = a_{i,j} \cdot C \cdot 2^{\lambda_i} \cdot 2^{\lambda_j}, \quad i, j \in [n],$$

$$\lambda_i = \begin{cases} 
((n-r) - i) \cdot \log_2(\alpha_{r,n}) & 1 \leq i \leq n-r \\
0 & n-r \leq i \leq r+1 \\
(i - (r+1)) \cdot \log_2(\alpha_{r,n}) & r+1 \leq i \leq n
\end{cases},$$

$$C = (\alpha_{r,n} - 1) \cdot \frac{1}{\alpha_{r,n}} = \frac{\alpha_{r,n} - 1}{(2r-n+2) - (2r-n) \cdot \alpha_{r,n}},$$

and $\alpha_{r,n}$ is the unique positive solution of the equation

$$\alpha_{r,n}^{n-r} + (2r-n) \cdot \alpha_{r,n} - (2r-n+2) = 0.$$
Finding $\alpha_{r,n}$ is more tricky

**Lemma**

Let $r = \rho \cdot (n-1)$, with $\frac{1}{2} < \rho < 1$ a constant. Then $\alpha_{r,n}$ is

$$\alpha_{r,n} = 1 + \left( \hat{t} + \Theta \left( \frac{1}{n} \right) \right) \left( 2^{\frac{1}{(n-1)(1-\rho)+1}} - 1 \right),$$

where

$$\hat{t} = \frac{1}{\ln(2)} \left( \frac{2(1-\rho)}{2\rho - 1} - W \left( \frac{(1-\rho) \exp \left( \frac{2(1-\rho)}{2\rho - 1} \right)}{2\rho - 1} \right) \right),$$

and where $W(\cdot)$ denotes Lambert's function, i.e., $W(\cdot)$ is defined by $z = W(z) \exp(W(z))$. 

---

$\ell_\infty$-Metric Permutation Balls  The $r = \Theta(n)$ Regime
$\hat{t}$ is linear?

![Graph showing the relationship between $\hat{t}$ and $\rho$]
The $r = \Theta(n)$ Regime
Theorem

Let \( r = \rho \cdot (n-1) \), with \( \frac{1}{2} < \rho < 1 \) a constant, then

\[
|B_{r,n}| \geq \frac{n^n \cdot 2^{t(2\rho-1)n} \cdot (1 - \rho)^n}{(et \ln(2))^n} \cdot 2^{o(n)}.
\]
And the lower bound is...

**Theorem**

Let $r = \rho \cdot (n-1)$, with $\frac{1}{2} < \rho < 1$ a constant, then

$$|B_{r,n}| \geq \frac{n^n \cdot 2^{t(2\rho-1)n} \cdot (1 - \rho)^n}{(e^t \ln(2))^n} \cdot 2^{o(n)}.$$

**Proof.**

$$\ln |B_{r,n}| \geq \ln(\text{per}(A_{r,n})) \geq \ln(n!/n^n) + \sum_{i,j \in [n]} \left( -q_{i,j}^* \ln \frac{q_{i,j}^*}{a_{i,j}} \right)$$

$$= -n - n \ln(\alpha_{r,n} - 1) + (n - r)(2r - n + 2) \ln(\alpha_{r,n}) + o(n).$$
And the lower bound is...(Cont.)

Proof.

We have:

\[-n \ln(\alpha_{r,n} - 1) = -n \ln \left( (\hat{t} + \Theta(n^{-1})) \left( 2^{\frac{1}{n-r}} - 1 \right) \right)\]

\[= -n \ln (\hat{t} + \Theta(n^{-1})) - n \ln \left( \frac{\ln(2)}{(1 - \rho)n} + O(n^{-2}) \right)\]

\[= -n \ln(\hat{t}) - n \ln \ln(2) + n \ln(1 - \rho) + n \ln(n) + o(n).\]

Similarly,

\[(n - r)(2r - n + 2) \ln(\alpha_{r,n}) = n(2\rho - 1)\hat{t} \ln(2) + o(n).\]

Combining both we get the result.
The new bounds reduces the gap

1. Asymptotic gap for off-the-shelf ball-size bounds
2. Sinkhorn-balancing bounds

Schwartz and Vontobel, to appear
Asymptotic gap for off-the-shelf ball-size bounds
Sinkhorn-balancing bounds
Bethe-permanent bounds

Schwartz and Vontobel, to appear
The code

2 Lower bound on [LTT'08] construction

3 Upper bound on [LTT'08] construction

4 GV lower bound on optimum code rate

5 Ball-packing upper bound on optimum code rate

6 Ball-packing upper bound with improved ball-size bound