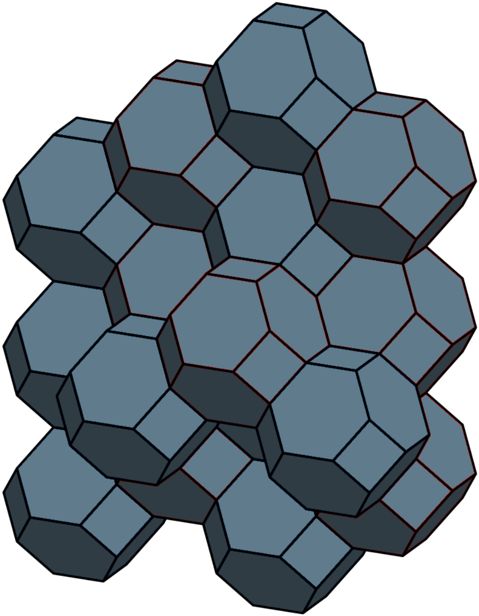
Balls of Permutations in the ℓ_∞-Metric

Moshe Schwartz

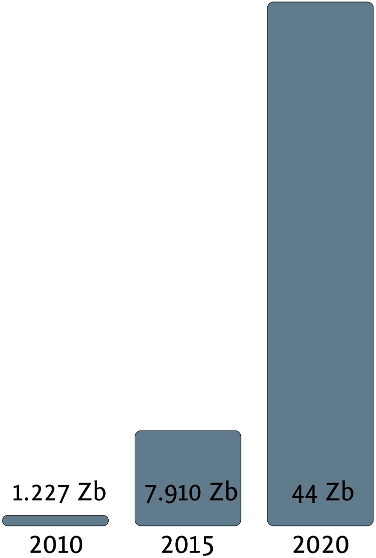
Electrical and Computer Engineering
Ben-Gurion University of the Negev



An age of data

Data storage is growing exponentially.

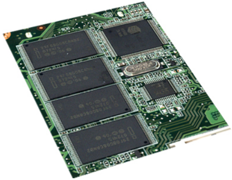
Storage technologies are crucial to enabling big data and cloud paradigms.



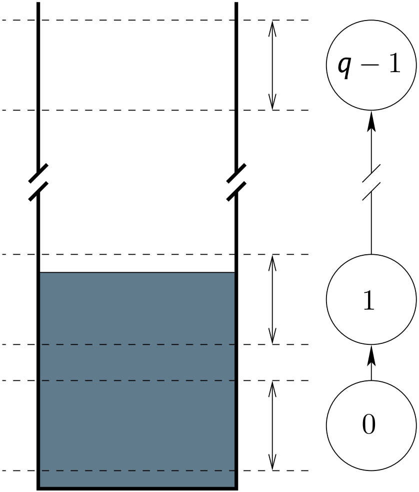
IDC Digital Universe Study, 2011 & 2014

What storage options do we have?

What storage options do we have?

Magnetic HDD	Flash SSD
	
Cheap Many R/W operations	
	Fast Energy Efficient Resilient (no mechanical parts)

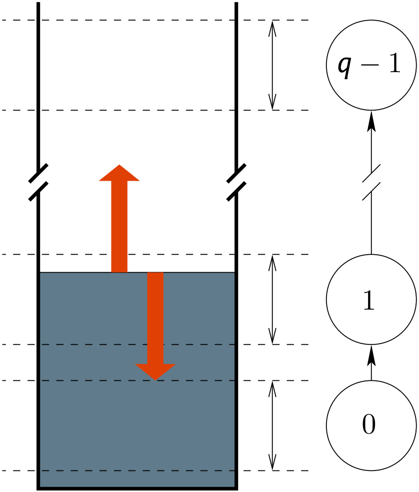
A flash memory cell is a bucket



Flash memory is an electronic non-volatile memory.

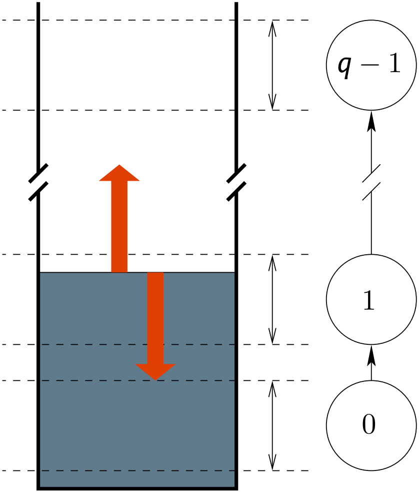
Conventional multi-level flash technology allows each cell to store q discrete values. These are represented by appropriate charge-level intervals.

Flash is an asymmetric noisy bucket



1. Adding charge is much easier than removing it. (asymmetric)
2. Errors commonly change charge level by a limited magnitude. (noisy)

Flash is an asymmetric noisy bucket

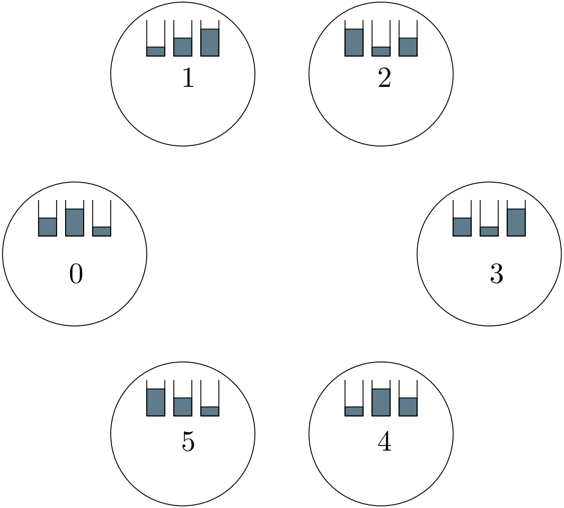


1. Adding charge is much easier than removing it. (asymmetric)
2. Errors commonly change charge level by a limited magnitude. (noisy)

$\Downarrow$

1. Writing is slow.
2. Data loss occurs with time.

Rank Modulation - Rethinking flash



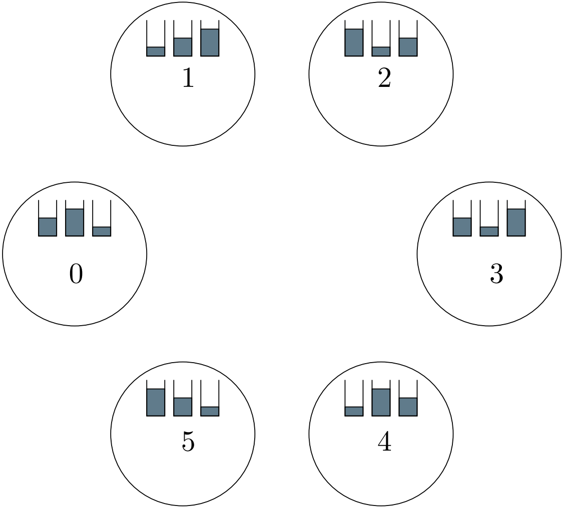
New Paradigm

Cells are compared against each other.

Information is stored in the permutation induced by their ranking.

Jiang, Mateescu, Schwartz, and Bruck, IEEE Trans. Inform. Th., 2009

Rank Modulation - Rethinking flash



New Paradigm

Cells are compared against each other.

Information is stored in the permutation induced by their ranking.

1. Faster writing
2. Less errors.

Jiang, Mateescu, Schwartz, and Bruck, IEEE Trans. Inform. Th., 2009

The goal

Construct Error-Correcting Codes
for
Rank Modulation

What do errors look like?

Limited-magnitude change in charge level

$\Downarrow$

Limited-magnitude change in ranks of cells

$\Downarrow$

Limited ℓ_∞-distance between stored permutations

Definition

For all $f, g \in S_n$, $d_\infty(f, g) = \max_i |f(i) - g(i)|$.

A simple code

(S_n, d_∞) is a metric space, so we can define error-correcting codes.

Definition

An (n, M, d)-LMRM code (Limited-Magnitude Rank-Modulation code) is a subset $C \subseteq S_n$ of size M such that $d_\infty(f, g) \geqslant d$ for all $f, g \in C$, $f \neq g$.

Tamo and Schwartz, IEEE Trans. Inform. Th., 2010
Kløve, Lin, Tsai, and Tzeng, IEEE Trans. Inform. Th., 2010

A simple code

(S_n, d_∞) is a metric space, so we can define error-correcting codes.

Definition

An (n, M, d)-LMRM code (Limited-Magnitude Rank-Modulation code) is a subset $C \subseteq S_n$ of size M such that $d_\infty(f, g) \geqslant d$ for all $f, g \in C$, $f \neq g$.

Construction

Given $n, d \in \mathbb{N}$ we construct

$$C = \{f \in S_n \mid \forall i \in [n], f(i) \equiv i \pmod{d}\}.$$

Tamo and Schwartz, IEEE Trans. Inform. Th., 2010

Kløve, Lin, Tsai, and Tzeng, IEEE Trans. Inform. Th., 2010

How good is the code?

We are interested in the asymptotics of the code. For an (n, M, d)-LMRM code we define the normalized distance and rate:

$$\delta = \frac{d}{n-1}, \qquad R = \frac{\log_2 M}{n}.$$

How good is the code?

We are interested in the asymptotics of the code. For an (n, M, d)-LMRM code we define the normalized distance and rate:

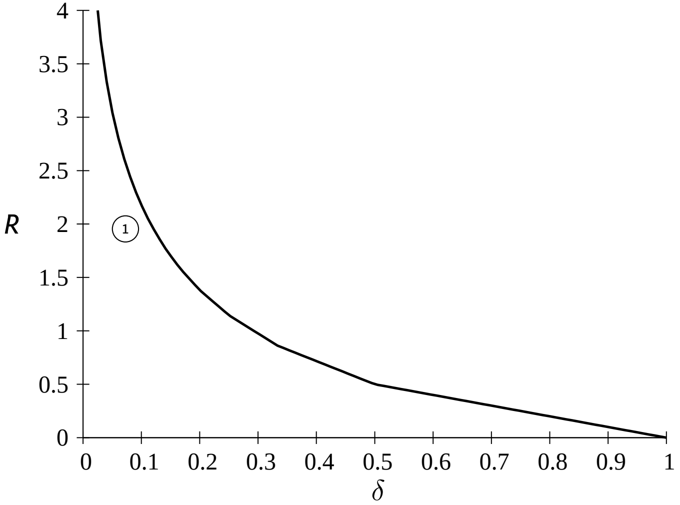
$$\delta = \frac{d}{n-1}, \qquad R = \frac{\log_2 M}{n}.$$

Theorem

For $0 < \delta \leqslant 1$, the construction produces codes of rate

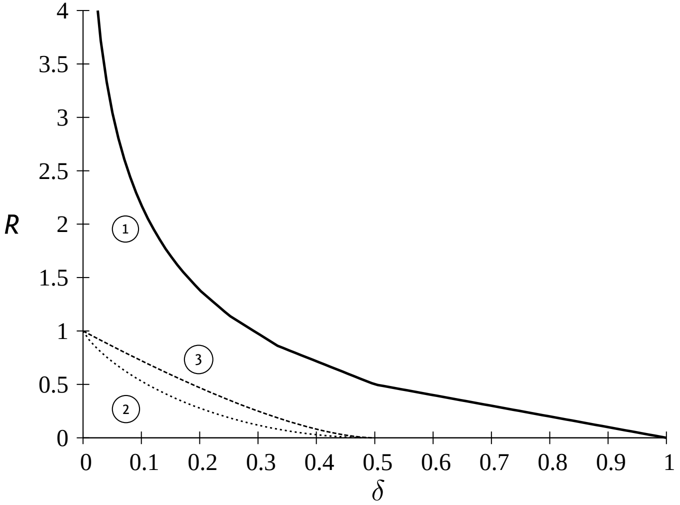
$$R = \left(1 - \delta \left\lfloor \frac{1}{\delta} \right\rfloor\right) \log_2 \left(\left\lceil \frac{1}{\delta} \right\rceil ! \right) + \left(\delta + \delta \left\lfloor \frac{1}{\delta} \right\rfloor - 1\right) \log_2 \left(\left\lfloor \frac{1}{\delta} \right\rfloor ! \right) + o(1).$$

How good is the code?



1 The code

How good is the code? (Cont.)



1. The code
2. Lower bound on [LTT'08] construction
3. Upper bound on [LTT'08] construction

Lin, Tsai, and Tzeng, Int. Symp. on Inform. Th., 2008

We can obtain better bounds

- A ball of radius r centered at f is $\mathcal{B}_{r,n}(f) = \{g \in S_n \mid d_\infty(f,g) \leqslant r\}$.

We can obtain better bounds

- A ball of radius r centered at f is $\mathcal{B}_{r,n}(f) = \{g \in S_n \mid d_\infty(f,g) \leqslant r\}$.
- The ℓ_∞-metric is right invariant, i.e., for all $f, g, h \in S_n$, we have $d_\infty(f,g) = d_\infty(fh, gh)$.

We can obtain better bounds

- A ball of radius r centered at f is $\mathcal{B}_{r,n}(f) = \{g \in S_n \mid d_\infty(f,g) \leqslant r\}$.
- The ℓ_∞-metric is right invariant, i.e., for all $f, g, h \in S_n$, we have $d_\infty(f,g) = d_\infty(fh, gh)$.
- Thus, ball sizes depend only on the radius and are denoted $|\mathcal{B}_{r,n}|$.

We can obtain better bounds

- A ball of radius r centered at f is $\mathcal{B}_{r,n}(f) = \{g \in S_n \mid d_\infty(f,g) \leqslant r\}$.
- The ℓ_∞-metric is right invariant, i.e., for all $f,g,h \in S_n$, we have $d_\infty(f,g) = d_\infty(fh,gh)$.
- Thus, ball sizes depend only on the radius and are denoted $|\mathcal{B}_{r,n}|$.

Theorem (Gilbert-Varshamov-Like Bound)

Let n, M, and d, be positive integers such that $\left|\mathcal{B}_{d-1,n}\right| M \leqslant n!$. Then there exists an (n,M,d)-LMRM code.

Theorem (Ball-Packing Bound)

Let C be an (n,M,d)-LMRM code. Then $\left|\mathcal{B}_{\lfloor (d-1)/2 \rfloor,n}\right| M \leqslant n!$.

But wait...

What is the size of a ball?

Permanants count certain permutation sets

Definition

For an $n \times n$ binary matrix $A = (a_{i,j})$, the permanent of A is defined as

$$\mathrm{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$

Permanants count certain permutation sets

Definition

For an $n \times n$ binary matrix $A = (a_{i,j})$, the permanent of A is defined as

$$\mathrm{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Permanants count certain permutation sets

Definition

For an $n \times n$ binary matrix $A = (a_{i,j})$, the permanent of A is defined as

$$\mathrm{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, f = [2, 1, 4, 3, 5] \text{ contributes } 1 \text{ to } \mathrm{per}(A).$$

Permanants count certain permutation sets

Definition

For an $n \times n$ binary matrix $A = (a_{i,j})$, the permanent of A is defined as

$$\mathrm{per}(A) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)}.$$

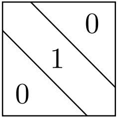
Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, f = [1, 4, 5, 3, 2] \text{ does not contribute to } \mathrm{per}(A).$$

The size of a ball is a permanent

Let us define the following family of $n \times n$ matrices, $A_{r,n} = (a_{i,j}^{r,n})$, where

$$a_{i,j}^{r,n} = \begin{cases} 1 & |i-j| \leqslant r, \\ 0 & \text{otherwise.} \end{cases}$$



Theorem

$|\mathcal{B}_{r,n}| = \mathrm{per}(A_{r,n})$.

Proof.

$f \in S_n$ contributes to $\mathrm{per}(A_{r,n})$ iff $|f(i) - i| \leqslant r$ for all i, which holds iff $d_\infty(f, \mathrm{Id}) \leqslant r$. □

Off-the-shelf bounds on permanents exist

Theorem (Brégman's Theorem)

Let A be an $n \times n$ binary matrix, with r_i 1's in the ith row. Then

$$\mathrm{per}(A) \leqslant \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}.$$

Theorem (Van der Waerden's Theorem)

Let A be an $n \times n$ doubly-stochastic matrix, then

$$\mathrm{per}(A) \geqslant \frac{n!}{n^n}.$$

Brégman, Soviet Math. Dokl., 1973
Falikman, Math. Zametki,, & Egorychev, Adv. in Math., 1981

Bound application

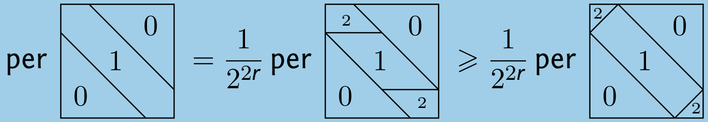
Brégman's Theorem gives an immediate upper bound:

$$|\mathcal{B}_{r,n}| = \mathrm{per}(A_{r,n}) \leqslant \begin{cases} ((2r+1)!)^{\frac{n-2r}{2r+1}} \prod_{i=r+1}^{2r} (i!)^{\frac{2}{i}} & 0 \leqslant r \leqslant \frac{n-1}{2}, \\ (n!)^{\frac{2r+2-n}{n}} \prod_{i=r+1}^{n-1} (i!)^{\frac{2}{i}} & \frac{n-1}{2} \leqslant r \leqslant n-1. \end{cases}$$

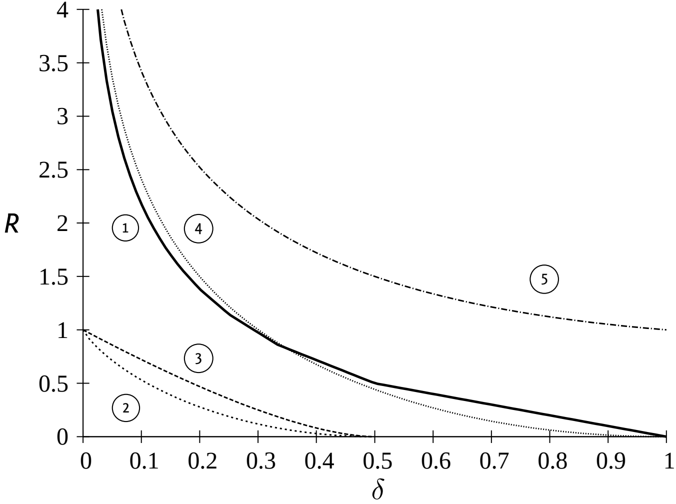
Van der Waerden's Theorem gives a lower bound:

$$|\mathcal{B}_{r,n}| = \mathrm{per}(A_{r,n}) \geqslant \begin{cases} \frac{n!(2r+1)^n}{2^{2r} n^n} & 0 \leqslant r \leqslant \frac{n-1}{2}, \\ \frac{n!}{2^{2(n-r-1)}} & \frac{n-1}{2} \leqslant r \leqslant n-1. \end{cases}$$

Proof Sketch



Adding the bounds to the picture



1. The code
2. Lower bound on [LTT'08] construction
3. Upper bound on [LTT'08] construction
4. GV lower bound on optimum code rate
5. Ball-packing upper bound on optimum code rate

A wide gap is the cause (in part)

Rewriting the bounds in asymptotic forms, we get

$$\left|\mathcal{B}_{\rho(n-1),n}\right| \leqslant \begin{cases} \left(\frac{2\rho n}{e}\right)^{n} \cdot \left(\frac{2}{e}\right)^{2\rho n} \cdot 2^{o(n)} & 0 \leqslant \rho \leqslant \frac{1}{2} \\ \frac{n^n}{e^{n(3-2\rho)}\rho^{2\rho n}} \cdot 2^{o(n)} & \frac{1}{2} \leqslant \rho \leqslant 1 \end{cases} := \Phi(\rho, n)$$

$$\left|\mathcal{B}_{\rho(n-1),n}\right| \geqslant \begin{cases} \frac{(2\rho n)^n}{2^{2\rho n} e^n} \cdot 2^{o(n)} & 0 \leqslant \rho \leqslant \frac{1}{2} \\ \frac{n^n}{e^n 2^{2n(1-\rho)}} \cdot 2^{o(n)} & \frac{1}{2} \leqslant \rho \leqslant 1 \end{cases} := \phi(\rho, n)$$

A wide gap is the cause (in part)

Rewriting the bounds in asymptotic forms, we get

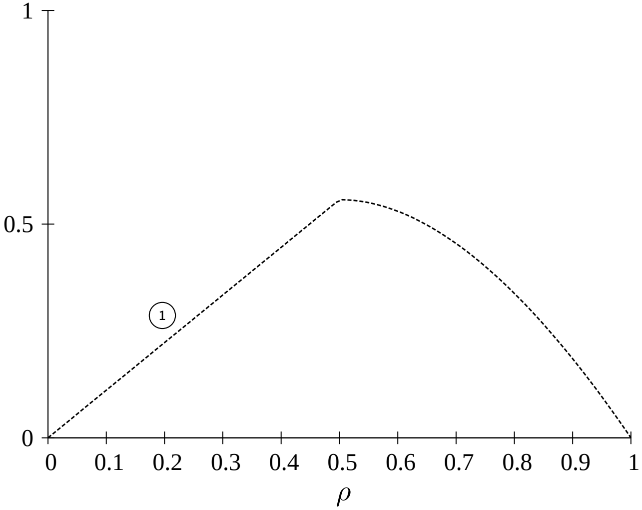
$$\left|\mathcal{B}_{\rho(n-1),n}\right| \leqslant \begin{cases} \left(\frac{2\rho n}{e}\right)^n \cdot \left(\frac{2}{e}\right)^{2\rho n} \cdot 2^{o(n)} & 0 \leqslant \rho \leqslant \frac{1}{2} \\ \frac{n^n}{e^{n(3-2\rho)}\rho^{2\rho n}} \cdot 2^{o(n)} & \frac{1}{2} \leqslant \rho \leqslant 1 \end{cases} := \Phi(\rho, n)$$

$$\left|\mathcal{B}_{\rho(n-1),n}\right| \geqslant \begin{cases} \frac{(2\rho n)^n}{2^{2\rho n} e^n} \cdot 2^{o(n)} & 0 \leqslant \rho \leqslant \frac{1}{2} \\ \frac{n^n}{e^n 2^{2n(1-\rho)}} \cdot 2^{o(n)} & \frac{1}{2} \leqslant \rho \leqslant 1 \end{cases} := \phi(\rho, n)$$

The asymptotic gap is therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 \frac{\Phi(\rho, n)}{\phi(\rho, n)} = \begin{cases} (4 - 2\log_2 e)\rho & 0 \leqslant \rho \leqslant \frac{1}{2}, \\ 2(1-\rho)(1-\log_2 e) - 2\rho \log_2 \rho & \frac{1}{2} \leqslant \rho \leqslant 1. \end{cases}$$

A wide gap is the cause (in part)



1 Asymptotic gap for off-the-shelf ball-size bounds

We can have zero gap for constant radii

The strategy

1. Show that permutations in a ball centered at the identity form a regular language.
2. Use Perron-Frobenius Theory to find the asymptotic size of the regular language.

Lehmer, Comb. Th. and its Applications, 1970
Schwartz, Linear Algebra and its Applications, 2009

We can have zero gap for constant radii

The strategy

1. Show that permutations in a ball centered at the identity form a regular language.
2. Use Perron-Frobenius Theory to find the asymptotic size of the regular language.

- The method is more general.
- For ease of presentation, only $(0, 1)$-matrices will be shown.

Lehmer, Comb. Th. and its Applications, 1970
Schwartz, Linear Algebra and its Applications, 2009

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

b_{-2}	b_{-1}	b_0	b_1	b_2
1	0	0	1	0

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

b_{-2}	b_{-1}	b_0	b_1	b_2
1	0	0	1	0

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

b_{-2}	b_{-1}	b_0	b_1	b_2
1	0	0	1	0

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

b_{-2}	b_{-1}	b_0	b_1	b_2
1	0	0	1	0

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b \in \{0, 1\}^{2r}$ indexed by $[-r, r-1]$.

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

b_{-2}	b_{-1}	b_0	b_1	b_2
1	0	0	1	0

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b \in \{0, 1\}^{2r}$ indexed by $[-r, r-1]$.
- At step i, $b_j = 1$ iff $i + j$ was already assigned.

Iteratively building permutations in $\mathcal{B}_{r,n}$

Example: For $n = 9$, $r = 2$, $i = 5$, the current status is

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$
3	2	1	6	?				

b_{-2}	b_{-1}	b_0	b_1	b_2
1	0	0	1	0

- At step $i \in [n]$ we set $f(i)$ by choosing a value from $i + [-r, r]$.
- We cannot choose a value that has been chosen before.
- We keep track of what has been chosen thus far by means of a binary string $b \in \{0, 1\}^{2r}$ indexed by $[-r, r - 1]$.
- At step i, $b_j = 1$ iff $i + j$ was already assigned.
- Moving from step i to step $i + 1$ we drop the left bit, and append a 0 rightmost bit.

More notation is needed

For a binary $b = b_1 b_2 \ldots b_m \in \{0,1\}^m$:

$b0$	Appending a 0 to the end of b
$\mathrm{wt}(b)$	The number of non-zero entries in b
$L(b)$	Erasing the leftmost bit, i.e., $b_2 \ldots, b_m$
e_k	The kth standard unit vector

We now build a finite state machine

Construction

We construct a digraph $\mathcal{G}_{[-r,r]} = (V, E)$ in the following way:

$$V = \left\{ b = b_{-r} b_{-r+1} \ldots b_{r-1} \in \{0,1\}^{2r} \mid \mathrm{wt}(b) = r \right\},$$

$$E = \left\{ b \xrightarrow{k} L(b0 + e_k) \mid b, L(b0 + e_k) \in V, \quad k \in T, \quad (b0)_k = 0 \right\}.$$

We now build a finite state machine

Construction

We construct a digraph $\mathcal{G}_{[-r,r]} = (V, E)$ in the following way:

$$V = \left\{ b = b_{-r}b_{-r+1}\dots b_{r-1} \in \{0,1\}^{2r} \mid \mathrm{wt}(b) = r \right\},$$
$$E = \left\{ b \xrightarrow{k} L(b0 + e_k) \mid b, L(b0 + e_k) \in V, \quad k \in T, \quad (b0)_k = 0 \right\}.$$

Theorem

There is a bijection between the set of permutations $\mathcal{B}_{r,n}$, and the paths of length n in $\mathcal{G}_{[-r,r]}$ starting and ending in the vertex 1^r0^r.

Exact asymptotic size

Theorem

For a constant r,

$$\lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{B}_{r,n}| = \log_2 \lambda(\mathcal{A}(\mathcal{G}_{[-r,r]})),$$

where $\mathcal{A}(\mathcal{G}_{[-r,r]})$ is the adjacency matrix of $\mathcal{G}_{[-r,r]}$, and $\lambda(\mathcal{A}(\mathcal{G}_{[-r,r]}))$ is the spectral radius of $\mathcal{A}(\mathcal{G}_{[-r,r]})$.

Exact asymptotic size

Theorem

For a constant r,

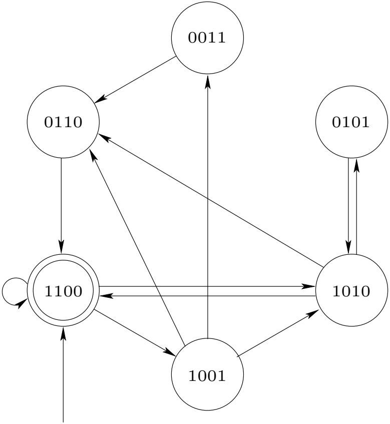
$$\lim_{n\to\infty} \frac{1}{n} \log_2 |\mathcal{B}_{r,n}| = \log_2 \lambda(\mathcal{A}(\mathcal{G}_{[-r,r]})),$$

where $\mathcal{A}(\mathcal{G}_{[-r,r]})$ is the adjacency matrix of $\mathcal{G}_{[-r,r]}$, and $\lambda(\mathcal{A}(\mathcal{G}_{[-r,r]}))$ is the spectral radius of $\mathcal{A}(\mathcal{G}_{[-r,r]})$.

Proof.

By the previous theorem, $|\mathcal{B}_{r,n}| = (\mathcal{A}(\mathcal{G}_{[-r,r]})^n)_{1,1}$. The matrix $\mathcal{A}(\mathcal{G}_{[-r,r]})$ has non-negative entries, and can be shown to be primitive. The claim follows immediately by Perron-Frobenius. □

Taking radius 2 as an example



Taking λ to be the largest (real) root of

$$\chi_{\mathcal{A}(\mathcal{G}_{[-2,2]})}(\lambda) = \lambda^6 - \lambda^5 - 2\lambda^4 - 2\lambda^3 - 2\lambda^2 + \lambda + 1,$$

we get

$$|\mathcal{B}_{2,n}| = \lambda^{n(1+o(1))} \approx 2.34148^n.$$

What about $r = \Theta(n)$?

Theorem

Let $M = (m_{i,j})$ be a $n \times n$ matrix with non-negative entries and $\mathrm{per}(M) > 0$. Additionally, we require that there are two diagonal matrices D and D' with positive diagonal elements such that $D \cdot M \cdot D'$ is a doubly-stochastic matrix. Let $Q = (q_{i,j})$ be an $n \times n$ doubly-stochastic matrix. Then

$$\log_2 \mathrm{per}(M) \geqslant \log_2 \frac{n!}{n^n} + \sum_{i,j \in [n]} \left(-q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}} \right).$$

Sinkhorn, Ann. Math. Stat., 1964

Using the Sinkhorn balancing

Proof.

$$\begin{aligned}
\log_2 \operatorname{per}(M) &= \log_2 \operatorname{per}(D \cdot M \cdot D') - \sum_{i\in[n]} \log_2(d_{i,i}) - \sum_{j\in[n]} \log_2(d'_{j,j}) \\
&\geqslant \log_2 \frac{n!}{n^n} - \sum_{i\in[n]} \log_2(d_{i,i}) - \sum_{j\in[n]} \log_2(d'_{j,j}) \\
&\geqslant \log_2 \frac{n!}{n^n} - \sum_{i\in[n]} \log_2(d_{i,i}) - \sum_{j\in[n]} \log_2(d'_{j,j}) \\
&\quad - \sum_{i,j\in[n]} q_{i,j} \log_2 \frac{q_{i,j}}{d_{i,i} \cdot m_{i,j} \cdot d'_{j,j}} \\
&= \log_2 \frac{n!}{n^n} - \sum_{i,j\in[n]} q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}} \qquad \square
\end{aligned}$$

We can find the optimal matrix

Lemma

Fix $\frac{n-1}{2} \leqslant r \leqslant n-2$. The optimal $Q = (q_{i,j})$ for $A_{r,n}$ is given by

$$q_{i,j} = a_{i,j} \cdot C \cdot 2^{\lambda_i} \cdot 2^{\lambda_j}, \quad i,j \in [n],$$

$$\lambda_i = \begin{cases} ((n-r)-i) \cdot \log_2(\alpha_{r,n}) & 1 \leqslant i \leqslant n-r \\ 0 & n-r \leqslant i \leqslant r+1 \\ (i-(r+1)) \cdot \log_2(\alpha_{r,n}) & r+1 \leqslant i \leqslant n \end{cases},$$

$$C = (\alpha_{r,n}-1) \cdot \alpha_{r,n}^{-(n-r)} = \frac{\alpha_{r,n}-1}{(2r-n+2)-(2r-n) \cdot \alpha_{r,n}},$$

and $\alpha_{r,n}$ is the unique positive solution of the equation

$$\alpha_{r,n}^{n-r} + (2r-n) \cdot \alpha_{r,n} - (2r-n+2) = 0.$$

Finding $\alpha_{r,n}$ is more tricky

Lemma

Let $r = \rho \cdot (n-1)$, *with* $\frac{1}{2} < \rho < 1$ *a constant. Then* $\alpha_{r,n}$ *is*

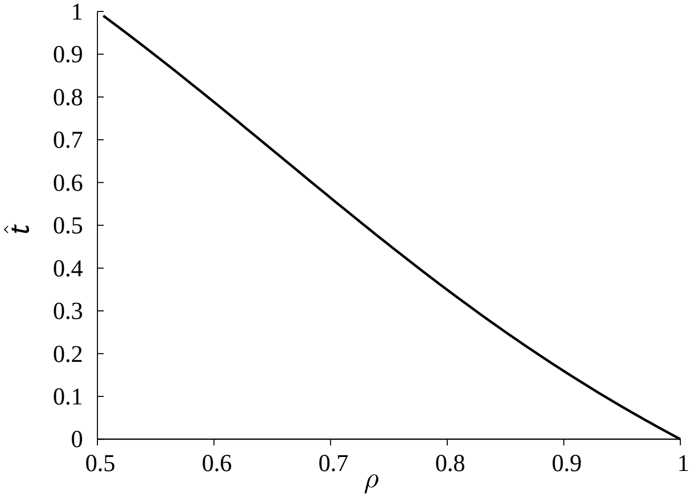
$$\alpha_{r,n} = 1 + \left(\hat{t} + \Theta\left(\frac{1}{n}\right)\right)\left(2^{\frac{1}{(n-1)(1-\rho)+1}} - 1\right),$$

where

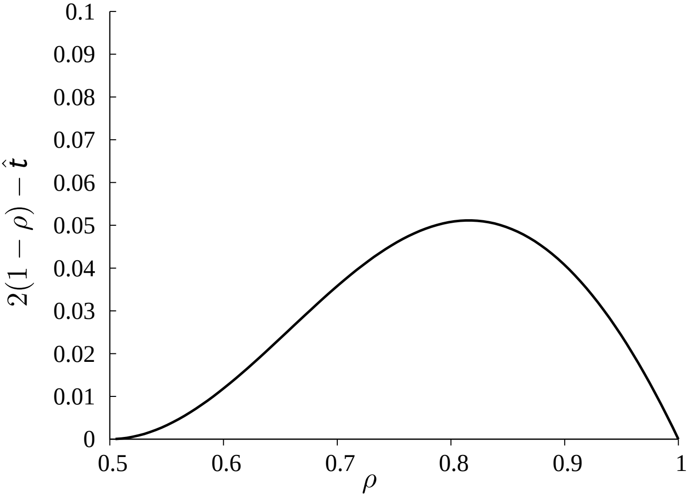
$$\hat{t} = \frac{1}{\ln(2)}\left(\frac{2(1-\rho)}{2\rho-1} - W\left(\frac{(1-\rho)\exp\left(\frac{2(1-\rho)}{2\rho-1}\right)}{2\rho-1}\right)\right),$$

and where $W(\cdot)$ *denotes Lambert's function, i.e.,* $W(\cdot)$ *is defined by* $z = W(z)\exp(W(z))$.

$\hat{t}$ is linear?



$\hat{t}$ is linear? Almost...



And the lower bound is...

Theorem

Let $r = \rho \cdot (n-1)$, *with* $\frac{1}{2} < \rho < 1$ *a constant, then*

$$|B_{r,n}| \geqslant \frac{n^n \cdot 2^{\hat{t}(2\rho-1)n} \cdot (1-\rho)^n}{(e\hat{t}\ln(2))^n} \cdot 2^{o(n)}.$$

And the lower bound is...

Theorem

Let $r = \rho \cdot (n-1)$, *with* $\frac{1}{2} < \rho < 1$ *a constant, then*

$$|B_{r,n}| \geqslant \frac{n^n \cdot 2^{\hat{t}(2\rho-1)n} \cdot (1-\rho)^n}{(e\hat{t}\ln(2))^n} \cdot 2^{o(n)}.$$

Proof.

$$\ln|B_{r,n}| \geqslant \ln(\mathrm{per}(A_{r,n})) \geqslant \ln(n!/n^n) + \sum_{i,j\in[n]} \left(-q^*_{i,j} \ln \frac{q^*_{i,j}}{a_{i,j}}\right)$$
$$= -n - n\ln(\alpha_{r,n} - 1) + (n-r)(2r-n+2)\ln(\alpha_{r,n}) + o(n).$$

And the lower bound is...(Cont.)

Proof.

We have:

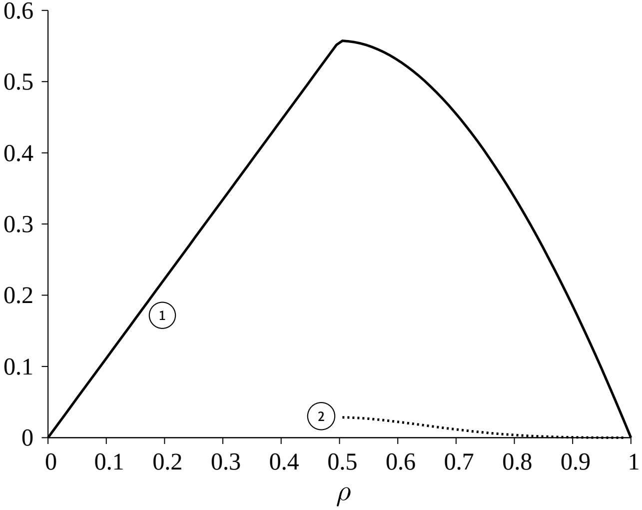
$$\begin{aligned} -n\ln(\alpha_{r,n}-1) &= -n\ln\left(\left(\hat{t}+\Theta\left(n^{-1}\right)\right)\left(2^{\frac{1}{n-r}}-1\right)\right) \\ &= -n\ln\left(\hat{t}+\Theta\left(n^{-1}\right)\right) - n\ln\left(\frac{\ln(2)}{(1-\rho)n}+O(n^{-2})\right) \\ &= -n\ln(\hat{t}) - n\ln\ln(2) + n\ln(1-\rho) + n\ln(n) + o(n). \end{aligned}$$

Similarly,

$$(n-r)(2r-n+2)\ln(\alpha_{r,n}) = n(2\rho-1)\hat{t}\ln(2)+o(n).$$

Combining both we get the result. □

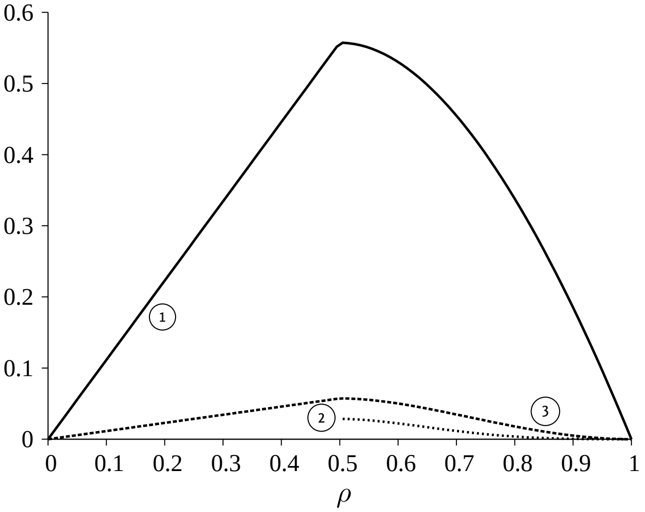
The new bounds reduces the gap



1. Asymptotic gap for off-the-shelf ball-size bounds
2. Sinkhorn-balancing bounds

Schwartz and Vontobel, to appear

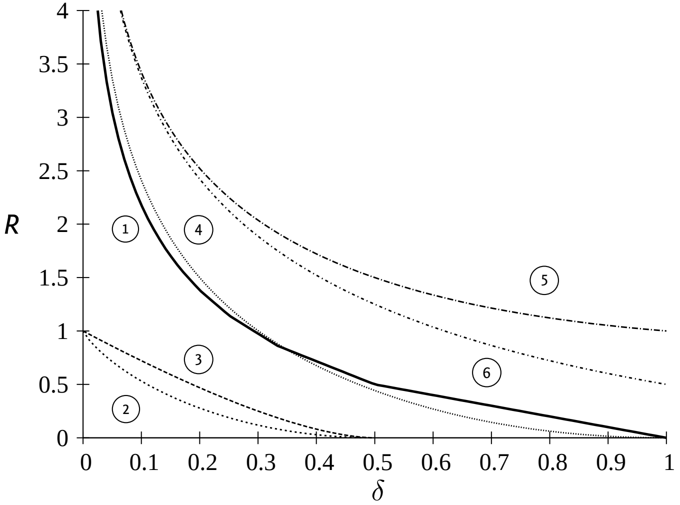
Wait, there's more...



1. Asymptotic gap for off-the-shelf ball-size bounds
2. Sinkhorn-balancing bounds
3. Bethe-permanent bounds

Schwartz and Vontobel, to appear

The final picture (for now)



1. The code
2. Lower bound on [LTT'08] construction
3. Upper bound on [LTT'08] construction
4. GV lower bound on optimum code rate
5. Ball-packing upper bound on optimum code rate
6. Ball-packing upper bound with improved ball-size bound