Lower Bounds for Locally Recoverable Codes

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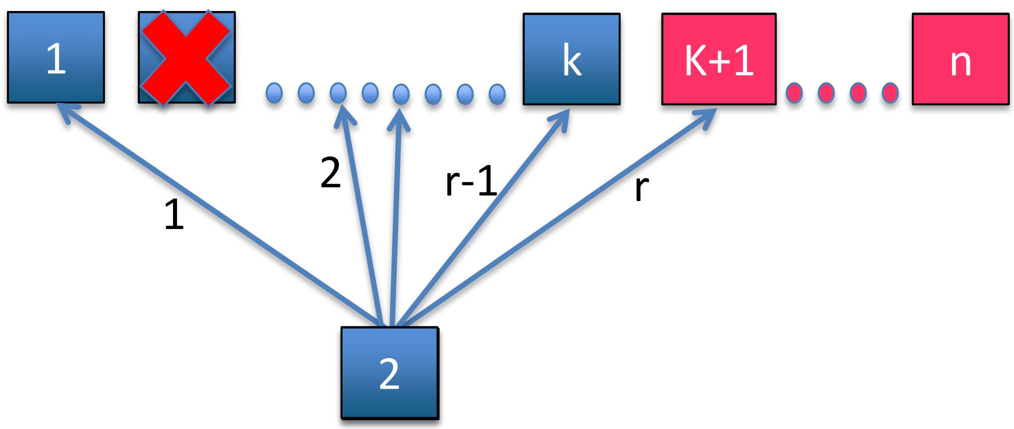
Coding Workshop - Technion

Outline

- Locally Recoverable (LRC) Codes
- Lower bound for LRC codes with large alphabet size
- Lower bound for LRC codes with constant alphabet size (GV-type bound)
- Summary

Locally Recoverable Codes (LRC)

- $(n, k, r) LRC$:
  - » Takes k blocks ⟹ produces n blocks
  - » <u>Any</u> block has a recovering set of r other blocks, $r \ll k$
  - » Clearly $1 \leq r \leq k$



Related Work

- *On the Locality of Codeword symbols - Gopalan, Huang, Simitci, and Yekhanin*
- Optimal Locally Repairable Codes - Rawat, Koyluoglu, Silberstein, Vishwanath,
- Optimal Locally Repairable Codes – T., Papaliopoulos, Dimkis
- Partial MDS codes - Blaum, Plank, Schwartz , Yaakobi
- Vijay Kumar’s group, Camilla Hollanti’s group, and Paul Siegel’s group
- And many more…

Locality and Minimum Distance

- $(n, k, r)LRC$:
  - ❖ $d \leq n - k + 2 - \lceil k/r \rceil$ *Gopalan, Huang, Simitci, and Yekhanin*
  - ❖ Observation: Smaller locality $\Longrightarrow$ lower failure resilience
  - ❖ Optimal $(n, k, r)LRC$ has minimum distance $d = n - k + 2 - \lceil k/r \rceil$
  - ❖ There exists an optimal (n, k, r) LRC over small finite fields *T. and Barg*
    - $|F| = n$
    - Generalization of RS Codes

LRC Codes and Multiple Recovering Sets

- (n, k, r, t) LRC :
  - Any symbol has t disjoint recovering sets of size r
  - $d \leq n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor$ *T. and Barg*

    Rawat, papaiopoulos, Dimakis and Vishwanath
  - $t = 0$: $d \leq n - k + 1$ Tight bound
  - $t = 1$: $d \leq n - k + 2 - \lceil k/r \rceil$ Tight bound
  - $t = 2$: $n - k - \left\lceil \frac{2k}{r-1} \right\rceil + 1 \leq d \leq n - \left(k - 1 + \left\lfloor \frac{k-1}{r} \right\rfloor + \left\lfloor \frac{k-1}{r^2} \right\rfloor\right)$
  - $t = 2$: $1 - R - \frac{2R}{r-1} \leq \delta \leq 1 - R - \frac{R}{r} - \frac{R}{r^2}$

LRC Codes and Multiple Recovering Sets

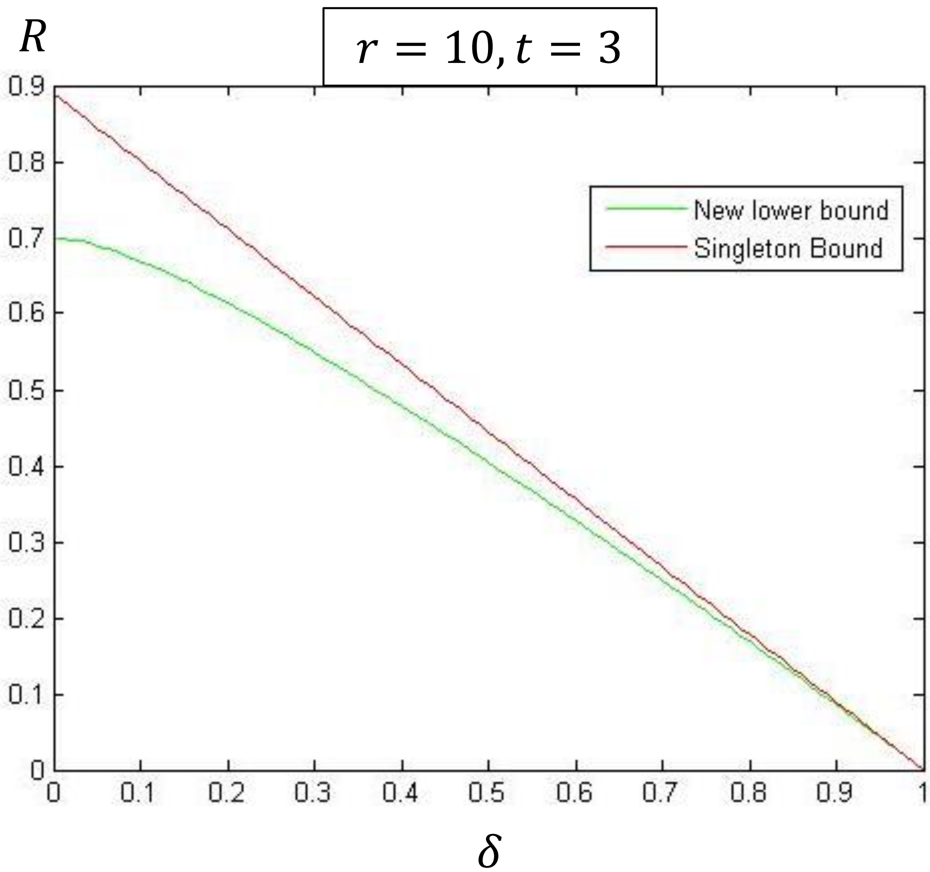
- <u>Theorem:</u> Let $R \leq 1 - \frac{t}{r+1}$ and δ that satisfies
  - $\delta = \frac{1 - \frac{t}{r+1} - R}{1 - t\gamma}$
  - $\frac{t-1}{t} h(\delta) - \frac{1}{r+1} h\big(\delta\gamma(r+1)\big) - \delta\gamma(r+1) h(\frac{1}{(\gamma(r+1)}) = 0$

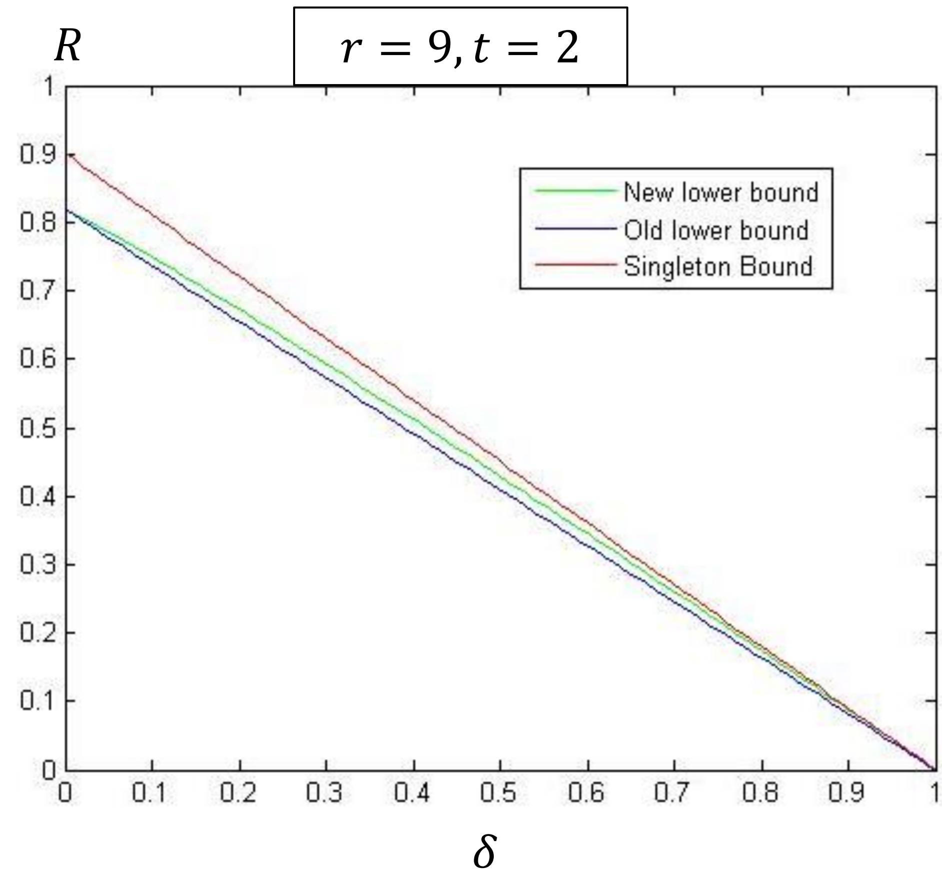
  then for sufficiently large alphabet there exists an (R, δ, r, t) LRC

- $R = rate$
- $\delta = distance$
- $r = size\ of\ recovering\ set$
- $t = \#\ recovering\ sets$

- <u>Remarks:</u>
  - Such $0 \leq \delta \leq 1$ always exists
  - This bound is better than all existing lower bounds
  - The proof uses the existence of graphs with good expansion properties (expander graphs)

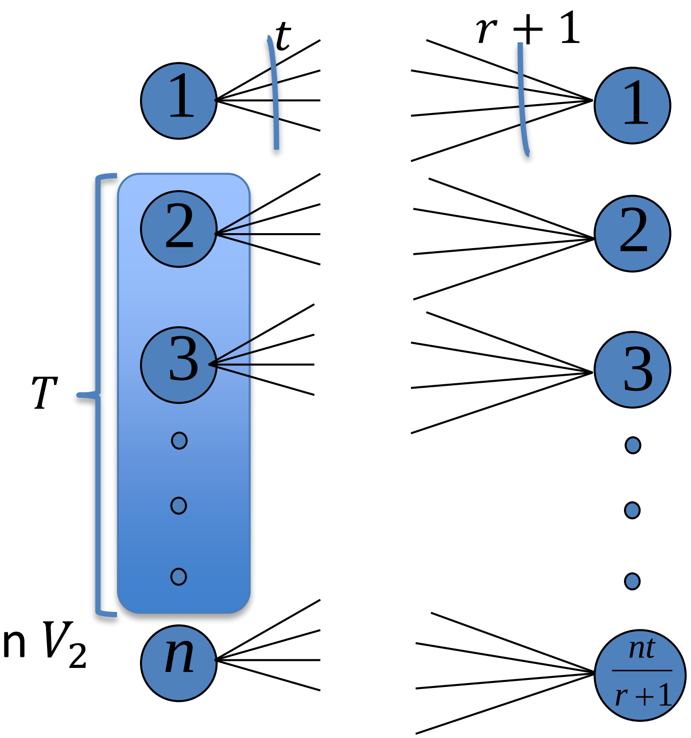
Some Plots


R
$r = 10, t = 3$
New lower bound
Singleton Bound
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
δ


R
$r = 9, t = 2$
New lower bound
Old lower bound
Singleton Bound
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
δ

From Expander Graphs to LRC Codes with Multiple Recovering Sets

- $G = (V = V_1 \cup V_2, E)$ a biregular graph:
  - $|V_1| = n$ and $\deg(v) = t$ for $v \in V_1$
  - $|V_2| = \frac{nt}{r+1}$ and $\deg(v) = r + 1$ for $v \in V_2$
  - G is $(\alpha, \gamma) - expander$ if every subset $\mathrm{T} \subset V_1, |T| \leq \alpha n$ has at least $\gamma t|T|$ neighbors in V_2



From Expander Graphs to LRC Codes with Multiple Recovering Sets

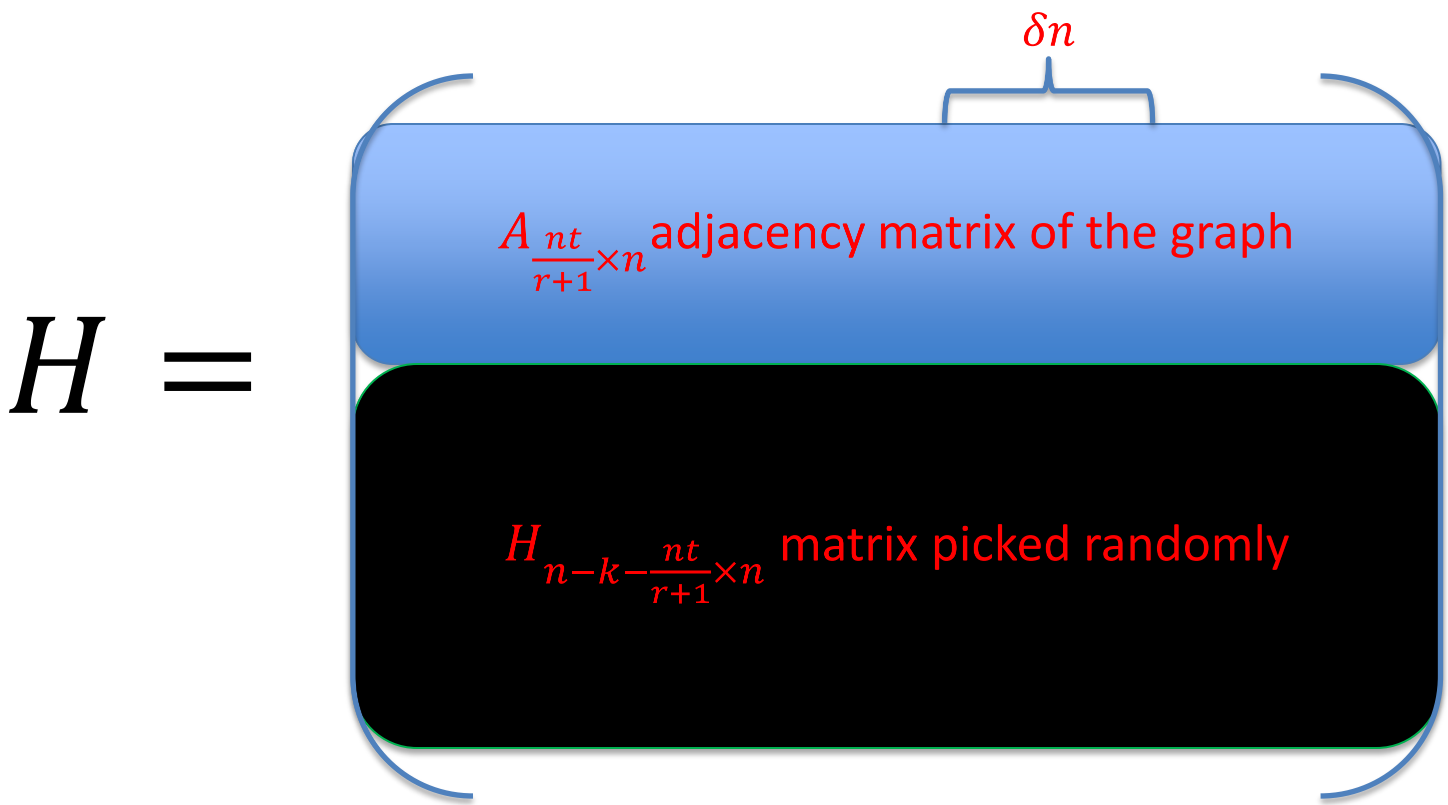
- The parity check matrix of the code:

$$H = \begin{pmatrix} A_{\frac{nt}{r+1}\times n} \text{ adjacency matrix of the graph} \\ H_{n-k-\frac{nt}{r+1}\times n} \text{ matrix picked randomly} \end{pmatrix}$$

- The nonzero entries of the adjacency matrix A are picked uniformly at random from the field F^*.

Why Expansion?

- Need to show that every δn columns are linearly independent



- Lemma: if the δn columns ``touch'' at least δn rows then w.h.p the columns are linearly independent.

Disjoint Recovering Sets

- Disjoint recovering sets $\leftrightarrow$ The girth of G is greater than 4
- Theorem (Bollobas): The number of cycles of length l in a uniformly picked regular graph behaves like a poisson r.v.
- Corollary: $\mathrm{P}(\text{girth of G} > 4) > \varepsilon$ for some $\varepsilon > 0$
- Theorem (Burstein and Miller): $\lim_{n\to\infty} \mathrm{P}(\text{G has "good" expansion}) = 1$
- Corollary: For large enough n there exists G with good expansion and girth > 4

GV-Type Bound for Constant Alphabet

Related Work

- Binary Cyclic Codes that are Locally Repairable - Goparaju and Calderbank (ISIT 2014)
- Optimal Linear and Cyclic LRC Codes over Small Fields – Zeh and Yaakobi (ITW 15)
- Achieving Arbitrary Locality and Availability in Binary Codes – Wang and Zhang
- An Upper Bound On the Size of LRC Codes – Cadambe and Mazumdar
  - Alphabet dependent bound on $(n, k, r)_q$ $LRC\ Code$

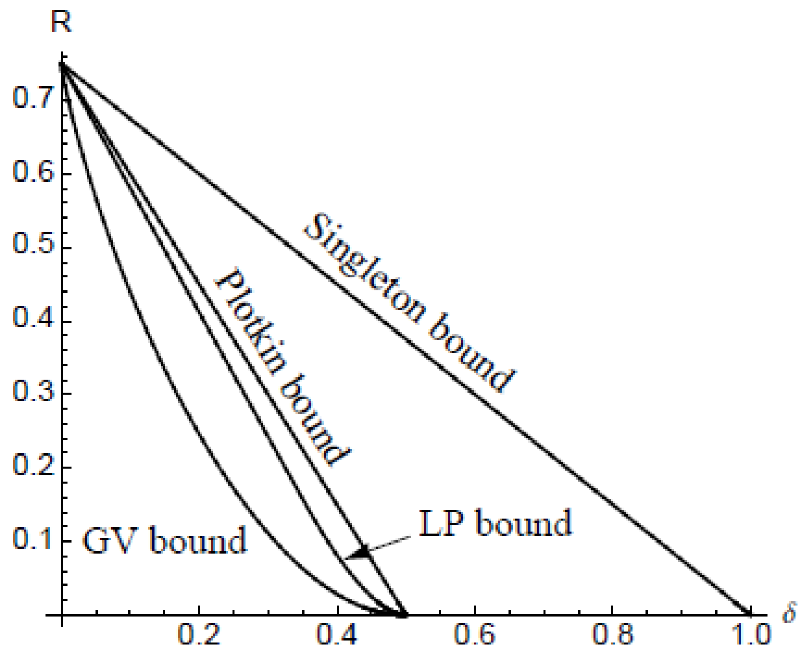
$$k \leq \min_{1 \leq m \leq \min\left(\left\lceil \frac{n}{r+1} \right\rceil, \left\lceil \frac{k}{r} \right\rceil\right)} \left\{ tr + k_q(n - m(r+1), d) \right\}$$

GV-Type Bound (Constant Alphabet)

- For $t = 1$ recovering set, the following asymptotic GV-type bound holds true:

1) $R_q(r,\delta) \geq \frac{r}{r+1} - min_{0<s\leq 1}\left\{\frac{\log_q b(s)}{r+1} - \delta \log_q s\right\}$

2) $b(s) = \frac{1}{q}\left((1+(q-1)s)^{r+1} + (q-1)(1-s)^{r+1}\right)$



(a) Bounds for codes with single recovering set.

GV-Type Bound (Constant Alphabet)

- Construct an $(n-k) \times n$ parity check matrix

$$H = \begin{bmatrix} H_U \\ H_L \end{bmatrix}, H_U = \begin{bmatrix} \boxed{H_0} & & & \\ & \boxed{H_0} & & \\ & & \ddots & \\ & & & \boxed{H_0} \end{bmatrix}$$

- H_U has $\frac{n}{r+1}$ copies of H_0

- H_0 is a simple parity check code of length $r+1$

- H_L is an $\left(\frac{nr}{r+1} - k\right) \times n$ matrix with entries picked uniformly and independently from F_q

GV-Type Bound (Constant Alphabet)

- $H \cdot \mathrm{x} = 0 \leftrightarrow H_U \cdot \mathrm{x} = 0$ and $H_L \cdot \mathrm{x} = 0$

- Upper bound the number of “bad” (low weight) vectors that satisfy $H_U \cdot \mathrm{x} = 0$ using the weight enumerator $b(s)$ of the code <u>checked</u> by H_U

$$H_U = \begin{bmatrix} \boxed{H_0} & & & \\ & \boxed{H_0} & & \\ & & \ddots & \\ & & & \boxed{H_0} \end{bmatrix}$$

- The weight enumerator of the code <u>generated</u> by H_U is easy to derive
- Using MacWilliams Identity derive the weight enumerator of the code with parity check matrix H_U:

$$b(s) = \frac{1}{q}((1+(q-1)s)^{r+1} + (q-1)(1-s)^{r+1})$$

GV-Type Bound (Constant Alphabet)

- Let M be the upper bound on the bad (low weight) vectors

$$M \geq |\{x : H_U \cdot x = 0, \mathrm{wt}(x) < \delta n\}|$$

- The entries of H_L are picked randomly $\rightarrow$

  for $\mathrm{x} \neq 0$ then $\mathrm{P}(H_L \cdot \mathrm{x} = 0) = \mathrm{q}^{-n(\frac{r}{r+1} - R)}$

- If $M \cdot \mathrm{q}^{-n\,(\frac{r}{r+1} - R)} < 1$ then there exists a parity check matrix $H = \begin{bmatrix} H_U \\ H_L \end{bmatrix}$ with minimum distance at least δn (union bound)

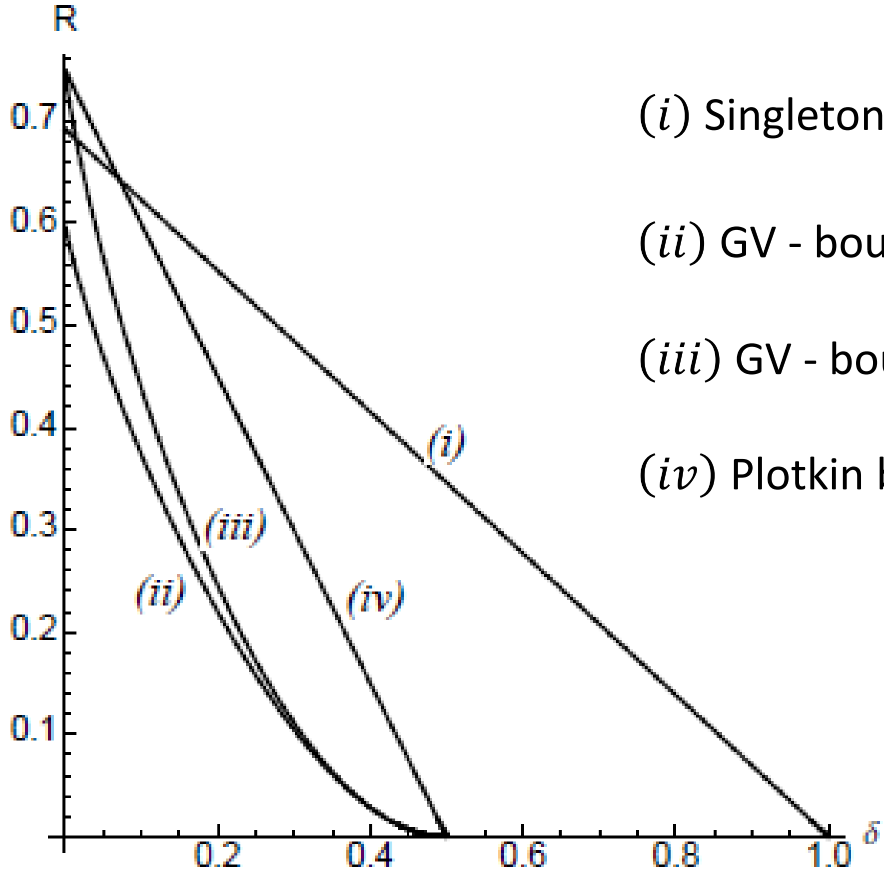
GV-Type Bound (Constant Alphabet) 2 Recovering sets

- For $t = 2$ recovering sets, the following asymptotic GV-type bound holds true:

$$1)\; R_q^{(2)}(r,\delta) \geq \frac{r}{r+2} - \min_{0<s\leq 1}\left\{\frac{1}{\binom{r+2}{2}}\log_q g_q^{(2)}(s) - \delta \log_q s\right\}$$

$$2)\; g_2^{(2)}(s) = \frac{1}{2^{r+2}}\sum_{i=0}^{r+2}\binom{r+2}{i}(1+s)^{\binom{r+2}{2}-i(r+2-i)}(1-s)^{i(r+2-i)}.$$

GV-Type Bound (Constant Alphabet) 2 Recovering sets



(b) Asymptotic bounds on codes with one and two recovering sets.

(i) Singleton Bound

(ii) GV - bound on $R_2^{(2)}(r, \delta)$

(iii) GV - bound on $R_2(r, \delta)$

(iv) Plotkin bound

GV-Type Bound (Constant Alphabet)

- Construct an $(n-k)\times n$ parity check matrix

$$H = \begin{bmatrix} H_U \\ H_L \end{bmatrix}, H_U = \begin{bmatrix} \boxed{H_0} & & & \\ & \boxed{H_0} & & \\ & & \ddots & \\ & & & \boxed{H_0} \end{bmatrix}$$

- H_U has $\frac{n}{\binom{r+2}{2}}$ copies of H_0

- H_0 is an $(r+1)\times\binom{r+2}{2}$ edge vertex incidence matrix of a complete graph K_{r+2} with one row deleted.
- H_L is an $\left(\frac{nr}{r+2}-k\right)\times n$ matrix with entries picked uniformly and independently from F_q

Summary

- Lower bounds for (n, k, r, t) $LRC\ code$ over large alphabet using expanders
  - Derive tighter Singleton-type bounds
  - Derandomize the construction using
    - Known expander graphs
    - Explicit assignment to the entries of the parity check matrix
- Lower bounds for $(n, k, r, t = 1{,}2)$ $LRC\ code$ with constant alphabet size (GV-type bound)

Thank you for listening