Codes for Partially Stuck-at Memory Cells

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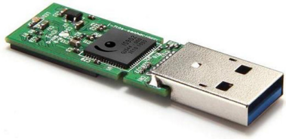
Workshop on Coding for Emerging Memories and Storage Technologies

Motivation: Non-volatile memories

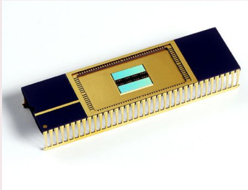


Multilevel flash memories

- electronic charge represents multiple levels
- if charge is trapped, level can only be increased
- degraded reliability: cells cannot/shouldn't represent high levels

Phase change memories (PCMs)

- multiple distinct physical states (one amorphous & some partially crystalline)
- cells might reach only crystalline states



Outline

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(Partially) Stuck-at Cells

Stuck-at Cells (memory with defects)

- binary cells: cell can be stuck-at level 0 or 1
- q-ary cells: cell can be stuck-at any level $s \in [q]$
- a stuck-at cell cannot change its level

Partially Stuck-at Cells

- q-ary cells: cell can be partially stuck-at any level $s \in [q]$
- a partially stuck-at cell can only store levels at least s

Known work:

- masking stuck-at cells first considered by [Kuznetsov-Tsybakov, 1974]
- stuck-at cells & random errors [Heegard, 1983], [Tsybakov, 1984], [Borden-Vinck, 1987]
- recently increased interested in stuck-at cells in PCMs [Lastras-Montano et al., 2010], [Kim-Kumar, 2013]

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Definitions

$(u, \mathbf{s})$-stuck-at-masking code ($(u, \mathbf{s})$-SMC) &
$(u, \mathbf{s})$-partially-stuck-at-masking code ($(u, \mathbf{s})$-PSMC)

A $\begin{cases} (u, \mathbf{s})\text{-SMC} \\ (u, \mathbf{s})\text{-PSMC} \end{cases}$ is a coding scheme with encoder $\mathcal{E}$ & decoder $\mathcal{D}$:

- $\mathcal{E}$: input: message m, locations & values (stored in $\mathbf{s} = (s_0, s_1, \ldots, s_{u-1})$) of u (partially) stuck-at cells

  output: vector $\mathbf{y}^{(m)}$ with $\begin{cases} y_i^{(m)} = s_i \\ y_i^{(m)} \geq s_i \end{cases}$ at stuck positions
- $\mathcal{D}$: input: $\mathbf{y}^{(m)}$

  output: message

Construction of $(u, \mathbf{s})$-SMCs

Theorem (Masking Stuck-At Cells, [Heegard, 1983])

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with minimum distance $d \geq u + 1$.
Then, there exists a $(u, \mathbf{s})$-SMC with redundancy $r = n - k$.

Example: $[7, 4, 3]_2$ Hamming code: $u = 2$ and $r = 3$ (store 4 bits)

- message $\mathbf{m} = (0, 1, 1, 0)$
- stuck positions: 1, 5
- define $\mathbf{w} = (0, 0, 0, \underbrace{0, 1, 1, 0}_{=\mathbf{m}})$

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

- find $\mathbf{z} \in \{0, 1\}^3$ s.t. $\mathbf{y} = \mathbf{w} + \mathbf{z} \cdot \mathbf{H}$ masks stuck-at cells
  $\implies \mathbf{z} = (1, 1, 0)$
- $\mathbf{y} = \mathbf{w} + \mathbf{z} \cdot \mathbf{H} = (0, 0, 0, 0, 1, 1, 0) + (1, 1, 0, 1, 1, 0, 0)$
  $= (1, 1, 0, 1, 0, 1, 0)$ ✓
- reconstruction possible since $(y_0, y_1, y_2) = \mathbf{z}$

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Outline

Bounds & Trivial Constructions of PSMCs

Theorem (Bounds on the Redundancy)

For any u partially stuck-at cells and levels $\mathbf{s} = (s_0, s_1, \ldots, s_{u-1})$, the minimum redundancy $r_q(n, u, \mathbf{s})$ to mask these cells satisfies

$$u - \log_q \left(\prod_{i=0}^{u-1} (q - s_i) \right) \leq r_q(n, u, \mathbf{s})$$
$$\leq \min \left\{ n \left(1 - \log_q \left(q - \max_i \{s_i\} \right) \right), \ \rho_q(n, u+1) \right\}.$$

- lower bound: stuck cells can represent $q - s_i$ levels, non-stuck cells q levels $\Rightarrow M \leq q^{n-u} \prod_{i=0}^{u-1} (q - s_i)$
- upper bound: use only levels $\max_i\{s_i\}, \ldots, q-1 \Rightarrow M = (q - \max_i\{s_i\})^n$
- upper bound: use $(u, \mathbf{s})$-SMC

Theorem (Improved Lower Bound for $s_i = s$)

For any (u, s)-PSMC, we have

$$r_q(n, u, s) \geq \log_q(u+1) - \log_q \left(1 + u(1 - s/q)^n \right).$$

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Construction for $u < q$

Theorem (Construction I for $u < q$)

If $u < q$ and $u \leq n$, then for all n, there exists an $(n, M = q^{n-1})_q$ $(u, \mathbf{1})$-PSMC with redundancy of one symbol.

Example: $u = 2$, $q = 3$, $n = 5$

- message $\mathbf{m} = (2, 0, 1, 0)$
- stuck positions: 1, 2
- define $\mathbf{w} = (0, 2, 0, 1, 0)$
- find z s.t. $\mathbf{y} = \mathbf{w} + z \cdot (1, 1, \ldots, 1)$ masks partially stuck-at cells:
  - $z = 0$: $\mathbf{y} = \mathbf{w} = (0, 2, 0, 1, 0)$ ↯
  - $z = 1$: $\mathbf{y} = \mathbf{w} + (1, 1, \ldots, 1) = (1, 0, 1, 2, 1)$ ↯
  - $z = 2$: $\mathbf{y} = \mathbf{w} + (2, 2, \ldots, 2) = (2, 1, 2, 0, 2)$ ✓
- reconstruction: $y_0 = z$ and thus, $\mathbf{y} - y_0 \cdot (1, 1, \ldots, 1)$ gives $\mathbf{w}$ and $\mathbf{m}$

$\Longrightarrow$ redundancy $r = 1$

$\Longrightarrow$ lower bound: $\max\{0.738, 0.787\} = 0.787$

$\Longrightarrow$ upper bound: $\min\{5 \cdot (1 - \log_3 2),\ \rho_3(5, 3) = 3\} = 1.845$

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Construction for $u < q$—Improvement

Theorem (Construction IB)

For any $u < q$, and $u \leq n$, there exists a $(u, \mathbf{1})$-PSMC over $[q]$ of length n and redundancy

$$r = 1 - \log_q \left\lfloor \frac{q}{u+1} \right\rfloor.$$

Example: $u = 2$ and $q = 6$, use same principle as before

- BUT: search $z \in \{0, 1, 2\}$ (instead of $z \in [q]$)
- store additional information in the redundancy cell by representing 0 by 0 and 3; 1 by 1 and 4; 2 by 2 and 5
- required redundancy: $1 - \log_6(2) \approx 0.613$ q-ary symbols

Theorem (Optimality of Construction IB)

If $(u+1)$ divides q, the $(u, \mathbf{1})$-PSMC from Construction IB is asymptotically optimal in terms of the redundancy.

(recall lower bound: $r_q(n, u, s) \geq \log_q(u+1) - \log_q \left(1 + u(1 - s/q)^n\right)$)

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Construction Based on q-ary Codes

Theorem (Construction II)

Let $u \leq q + d - 3$, $u \leq n$, $k < n$, and let $\mathbf{H}$ be the $(n-k) \times n$ parity-check matrix of an $[n, k, d]_q$ code. Then, there exists a $(u, \mathbf{1})$-PSMC over $\mathbb{F}_q$ of length n and redundancy $r = n - k$.

Corollary

If q is a prime power, then for all $u \leq n$, there exists a $(u, \mathbf{1})$-PSMC with redundancy $r = \max\{1, \rho_q(n, u - q + 3)\}$.

Example: $u = 5$, $q = 5$ and $n = 30$.

- our construction: $r = \rho_q(n, u - q + 3) = \rho_5(30, 3) = 3$
- lower bound: $\max\{0.693, 1.11\} = 1.11$
- upper bound: $\min\{4.16, \rho_5(30, 6) = 8\} = 4.16$

Construction Based on Binary u-SMCs

Theorem (Construction III Using Binary Codes)

Let n, $q \geq 4$ and $u \leq n$ be positive integers and let $\widetilde{u} = \lfloor 2u/q \rfloor$. Assume that $\widetilde{\mathcal{C}}$ is an $[n, k, d \geq \widetilde{u}+1]_2$ binary $(\widetilde{u}, \widetilde{\mathbf{s}})$-SMC. Then, there exists a $(u, \mathbf{1})$-PSMC over $[q]$ of length $(n+1)$ and redundancy

$$r = (n-k-1)\log_q\left(\frac{q}{\lfloor q/2 \rfloor}\right) + 2.$$

Example (simplified): Let $u = q = 3$, $n = 8$

- partially stuck positions: 0, 5, 6
- message $\mathbf{m} = (1, 2, 2)$, define $\mathbf{w} = (0, 0, 0, 0, 0, 1, 2, 2)$
- add a codeword of a binary SMC to mask the partially stuck-at cells: $\mathbf{w} = (0, 0, 0, 0, 0, 1, 2, 2) \Rightarrow$ use binary SMC with $\widetilde{u} = 2$
- Use parity-check matrix of an $[8, 4, 4]_2$ code:

$$\mathbf{y} = \mathbf{w} + (1, 0, 1, 0) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} = (1, 0, 1, 0, 1, 1, 2, 0) \checkmark$$

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Codes for Unreachable Levels

$(u, \mathbf{s})$-unreachable-masking code

A $(u, \mathbf{s})$-UMC is a coding scheme with encoder $\mathcal{E}$ and decoder $\mathcal{D}$:

- $\mathcal{E}$: input is message m, locations & values of u stuck-at cells,
  output is $\mathbf{y}^{(m)}$ with $y_i^{(m)} \leq s_i$ at unreliable positions
- $\mathcal{D}$: input is $\mathbf{y}^{(m)}$,
  output is message

Theorem

Given a $(u, \mathbf{s})$-PSMC with redundancy r and partially stuck levels $\mathbf{s} = (q-1-s_0, q-1-s_1, \dots, q-1-s_{u-1})$.
Then, this code can be used as a $(u, \widetilde{\mathbf{s}})$-UMC with redundancy r and levels $\widetilde{\mathbf{s}} = (s_0 \ s_1 \ \dots \ s_{u-1})$.

Outline

Capacity & Maximum Code Rate

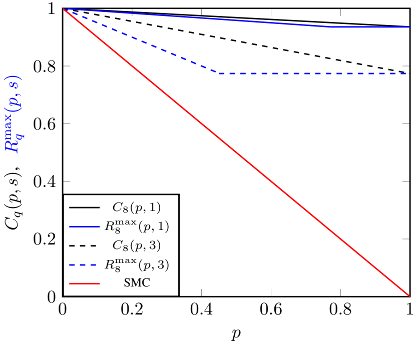
Theorem

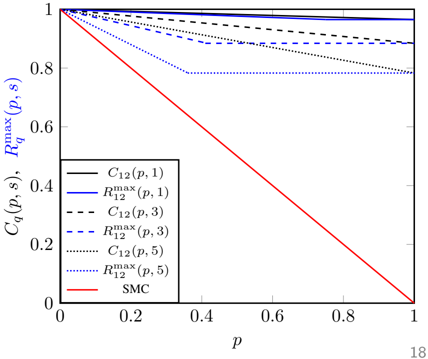
The capacity of the q-ary partially stuck-at level s channel is

$$C_q(p,s) = 1 - p\log_q\left(\frac{q}{q-s}\right),$$

where p is the probability that a cell is partially stuck-at level s.

$$R_q^{\max}(p,s) = \begin{cases} 1 - \frac{2sp}{q}\log_q(s+1) & \text{if } p \leq \frac{q}{2s}\log_{s+1}\left(\frac{q}{q-s}\right) \\ \log_q(q-s) & \text{else} \end{cases}$$





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Overview of our Constructions

Partially stuck-at level 1:

	Upper bound on $r_q(n, u, \mathbf{1})$
$u < q$	$1 - \log_q \left\lfloor \frac{q}{u+1} \right\rfloor$
$u \geq q$	$\rho_q(n, u - q + 3)$
any u, q	$\left(\rho_q\left(n, \lfloor \frac{2u}{q} \rfloor + 1\right) - 1\right) \cdot \log_q \left(\frac{q}{\lfloor q/2 \rfloor}\right) + 2$

Generalized levels $\mathbf{s} = (s_0, s_1, \ldots, s_{u-1})$:

	Upper bound on $r_q(n, u, \mathbf{s})$
$\sum_{i=0}^{u-1} s_i < q$	$r = 1 - \log_q \left\lfloor \frac{q}{\sum_{i=0}^{u-1} s_i + 1} \right\rfloor$
$u \geq \left\lceil \frac{q}{\max_i s_i} \right\rceil$	$\rho_q\left(n, u - \left\lceil \frac{q}{\max_i s_i} \right\rceil + 3\right)$
any u, q	$\left(\rho_q\left(n, \left\lfloor \frac{\sum_{i=1}^{q-1} u_i \cdot \sigma_i}{q} \right\rfloor + 1\right) - 1\right) \cdot \log_q \left(\frac{q}{\lfloor q/Q \rfloor}\right) + 2$, where $Q \geq \max_i\{s_i\} + 1$ is a prime power and $\sigma_i = \min\{q, Q + i - 1\}$, $\forall i \in [q]$.

Conclusion

Our Contribution

- new model of partially stuck-at memory cells
- lower & upper bounds on the redundancy of PSMCs
- three constructions for different ranges of parameters
- new codes for cells with unreliable levels
- capacity analysis

Outlook

- better code constructions
- combination with additional random errors

Thank you! Questions?

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Generalization to Arbitrary Levels

Theorem (Generalized Construction I)

Let u and n be positive integers and assume that $u \leq n$ cells are partially stuck-at levels $\mathbf{s} = (s_0, s_1, \ldots, s_{u-1}) \in \{1, \ldots, q-1\}^u$. If $\sum_{i=0}^{u-1} s_i < q$, then there exists a $(u, \mathbf{s})$-PSMC over $[q]$ with redundancy

$$r = 1 - \log_q \left\lfloor \frac{q}{\sum_{i=0}^{u-1} s_i + 1} \right\rfloor .$$

However, for general $\mathbf{s}$, it is not clear if this construction is asymptotically optimal...

Construction Based on Binary u-SMCs—Example (i)

Example: $n = 15$, $q = 4$, $u = 5$ and $U = \{1, 4, 8, 12, 15\}$.
$\Longrightarrow \widetilde{u} = \lfloor 2u/q \rfloor = 2$, use $[15, 11, 3]_2$ code $\widetilde{\mathcal{C}}$ as binary 2-SMC with

$$\mathbf{H} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&1&1&1&1&1&1 \\ 0&1&0&0&0&0&1&1&1&0&0&0&1&1&1 \\ 0&0&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 0&0&0&1&0&1&1&0&1&1&0&1&0&1&1 \end{pmatrix}.$$

- messages $\mathbf{m} = (0, 3, 2, 1, 2, 2, 3, 1, 3, 2, 2) \in [3]^{11}$, $\mathbf{m}' = (1, 0, 1) \in [1]^3$
- define $\mathbf{w} = (0, 0, 0, 0, \underbrace{0, 3, 2, 1, 2, 2, 3, 1, 3, 2, 2}_{\mathbf{m}}, 0)$
- find $z \in [q]$ s.t. the number of 0s/3s is minimal in $(\mathbf{w}^{(z)})_U = (\mathbf{w} + z \cdot (1, 1, \dots, 1))_U$:
  $\Longrightarrow z = 1$ and $\mathbf{w}^{(z)} = (1, 1, 1, 1, 1, 0, 3, 2, 3, 3, 0, 2, 0, 3, 3, 1)$
- mask the 2 usual stuck cells in $\mathbf{v} = (0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0)$ with binary 2-SMC:
  $\widetilde{\mathbf{c}} = ((1, 0, 0, 0) \cdot \mathbf{H}, 0) = (1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0)$
- encode additional message: $\widetilde{\mathbf{m}} = (2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
- store $\mathbf{y} = \mathbf{w}^{(z)} + \widetilde{\mathbf{c}} + \widetilde{\mathbf{m}} = (0, 1, 3, 1, 1, 0, 3, 2, 3, 0, 1, 3, 1, 0, 0, 1)$

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Construction Based on Binary u-SMCs—Example (ii)

Decoding process:

given $\mathbf{y} = (0,1,3,1,1,0,3,2,3,0,1,3,1,0,0,1)$

- recover z: $y_3 - y_{15} = 0 \Longrightarrow \widehat{z} = 1$
- $\widehat{\mathbf{y}} = \mathbf{y} - \widehat{z} \cdot (1,1,\ldots,1) = (3,0,2,0,0,3,2,1,2,3,0,2,0,3,3,0)$
- $\widehat{\mathbf{m}'} = (\lfloor \widehat{y}_0/2 \rfloor, \ldots, \lfloor \widehat{y}_{n-k-2}/2 \rfloor) = (\lfloor 3/2 \rfloor, 0, \lfloor 2/2 \rfloor) = (1,0,1)$
- $\widehat{\mathbf{t}} = (\widehat{y}_0 - 2\widehat{m}'_0, \ldots, \widehat{y}_{n-k-2} - 2, \widehat{m}'_{n-k-2}, \widehat{y}_{n-k-1}) \bmod q =$ $(3-2, 0-0, 2-2, 0) = (1,0,0,0)$
- $\widehat{\mathbf{c}'} = \widehat{\mathbf{t}} \cdot \mathbf{H} = (1,0,0,0,0,0,0,0,0,1,1,1,1,1,1)$
- $\widehat{\mathbf{m}} = (0,3,2,1,2,2,3,1,3,2,2).$

The redundancy to mask these five partially stuck-at-1 cells is therefore $r = 3.5$ q-ary cells.